

Integration of time-varying cocyclic one-forms against rough paths

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Abstract

We embed the rough integration in a larger geometrical/algebraic framework of integrating one-forms against group-valued paths, and reduce the rough integral to an inhomogeneous analogue of the classical Young integral. We define dominated paths as integrals of one-forms, and demonstrate that they are stable under basic operations.

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1 Introduction

1.1 Overview

In the early days of rough paths theory, and in the earlier work of Young [31], it was understood that there is a natural interplay between Lipschitz functions (or one-forms) and rough paths, with the integration being the intermediary. There are a number of characterizations for the Banach space of Lipschitz functions of degree $\gamma > 0$, and in this article we particularly take the view of Stein [30], that if \mathcal{K} is a subset of an affine space \mathcal{S} then a Lipschitz function on \mathcal{K} of degree γ is a continuous map $f : x \mapsto p(x)(\cdot)$ taking values in polynomial functions¹ on \mathcal{S} of degree strictly less than γ . The Lipschitz degree of f describes the varying speed of the polynomials: the higher the Lipschitz degree the slower the changing speed of the polynomials. The idea is that f prescribes a consistent family of global functions which are tangent to the trace $\{p(x)(x) | x \in \mathcal{K}\}$. It is the polynomials, the norms on them, and interactions between them that are critical, and in general the mapping $x \mapsto p(x)(x)$ does not carry nearly enough information about f unless \mathcal{K} is open.

A key point about Lipschitz functions we use here is somehow counter-intuitive. If \mathcal{K} is bounded, open, and connected, and p is a polynomial function, then one should think of the function $f : x \mapsto p(\cdot)$ as a *constant* Lipschitz function. Polynomial functions are in this sense constant functions in the space of Lipschitz functions. The view of polynomials as basic ingredients in the larger space of Lipschitz functions, or more generally the view of closed (cocyclic) one-forms as the basic ingredients in the space of time-varying one-forms, is at the heart of the framework introduced in this paper.

In particular, we reinterpret the rough line integral as the integral of a slowly-varying polynomial one-form against a rough path, where there is neither a given point nor a power series associated with a polynomial, and the customary view as a power series around a point on the path somewhat clouds this understanding because of the erratic movement of the point as the path evolves.

The original integration in the theory of rough paths defines an integral

$$\int_{u \in [s, t]} \alpha(X_u) dX_u$$

for a $\text{Lip}(\gamma)$ one-form α against a p -rough path X for $p < \gamma + 1$. A crucial restrictive assumption was that the one-form α depended on X_u but not on u directly. The assumption, standard in the Itô theory, that $\alpha \in L^2(du)$ is far too permissive for a deterministic theory. On the other hand, it was shown in [25] that if X is a p -rough path and h is a continuous path of finite q -variation then (X, h) is canonically a rough path providing $p^{-1} + q^{-1} > 1$ (see also [22, 13, 17]). Letting $h(t) = t$ allows one to consider

$$\int_{u \in [s, t]} \alpha(X_u, u) dX_u$$

with appropriate smoothness assumptions on α . L. G. Gyurko, in his thesis, gave mixed smoothness conditions that ensure the integral is well defined. In this paper we introduce a geometrically richer class of integrands (needed to get the algebraic closure) and provide a stronger and more intrinsic approach to the integration of one-forms.

Starting with a geometric rough path X , one sees from [14] (equation (18), and the next section about integration) that a path controlled by X is a path Y that can formally be identified with the integral of a time varying one-form G against X and modified by (or identified up to) a path of bounded q -variation where $p^{-1} + q^{-1} > 1$. The smoothness condition assumed for G in [14] is less restrictive than ours and does not guarantee the existence of the integral against X . On the other hand, geometrically the condition on G in [14] is more restrictive because G is a one-form on the base Banach space whereas we consider one-forms on the group. This allows us to prove an algebra property for integrals and also makes the dominated rough path a function of the integrand giving a simple linear parameterisation of the space of controlled rough paths. We could also consider the semi-martingale like spaces when one adds a q -variation perturbation. All integrals make sense as was observed in [25] page 259. However, it seems interesting to understand the integrals as a class in its own right before considering such perturbations. This way we introduce a natural and approximately dense space of functions on rough path space. Adding the q -variation perturbations would remove that clarity.

The initial goal of the theory of rough paths is to tackle the non-closability of the integral map for paths of low regularity. Lyons [25] observed that the integral map becomes continuous (and so closable),

¹A polynomial function of degree (at most) n is a globally defined function whose $(n + 1)^{\text{th}}$ derivative exists and is identically zero.

if one lifts the original integral in a Banach space to a consistent integral in a topological group. This lift is essentially nonlinear due to the nonlinearity of the group and is provably necessary even to have an integral defined deterministically for almost all Brownian sample paths. The integrals of a fixed rough path are jointly a rough path so collectively they have a linear structure, which is important in the proof of the unique existence of solutions to rough differential equations [25]. The linear structure is captured in a beautiful way by Gubinelli [14, 15] and he defined weakly controlled paths as a class of paths whose local behavior is comparable to a given rough path. For a fixed reference rough path, the space of controlled paths is linear, and there exists a canonical enhancement of a controlled path to a group-valued path (when $2 \leq p < 3$). The linearity of space and the existence of canonical enhancement are nice properties that general rough paths can not have, and they give considerable convenience e.g. when solving a rough differential equation. In [15], Gubinelli defined branched rough paths, and established the relationship between the evolution of a branched rough path and the Connes-Kremer Hopf algebra [6] (see also Butcher group [2]). He defined weakly controlled paths for branched rough paths, and defined the integration of a weakly controlled path for $p \geq 1$. More recently, Friz and Hairer [10] summarized key theorems in the theory of rough paths by employing Gubinelli's approach, and combined it with a brief introduction to the recent breakthrough made by the theory of regularity structures [16]. In particular they defined controlled rough paths as functions taking values in tensor algebra and defined the integration of a controlled path accordingly. The theory of rough paths has a wealth of literature, and there are many other formulations, e.g. [7, 12, 8, 19] etc. For a range of more detailed expositions, see [23, 20, 21, 26, 13].

We used the graded algebraic structure in proving the existence of the integral and in defining the set of dominated paths, so our setting is not far from the tensor algebra used in [25, 14] and the Connes-Kremer Hopf algebra used in [15]. The barrier between tensor algebra and Connes-Kremer Hopf algebra is not rigid. In [17] Hairer and Kelly proved that branched rough paths can equally be defined as Hölder paths taking values in some Lie group, and that every branched rough path can be encoded in a geometric rough path via a graded morphism of Hopf algebras so that solving a differential equation driven by a branched rough path is equivalent to solving an extended differential equation driven by a geometric rough path. In this paper, we identify structural properties of a Banach algebra and its associated topological group that enable basic operations, and construct an algebraic framework that subsumes tensor algebra and Connes-Kremer Hopf algebra. The theory of rough paths provides a natural framework to integrate group-valued paths, and is the incentive of this paper.

Popular approaches to rough integration use a representation of the group in the truncated tensor algebra to linearize the group-valued path, and treat a rough path as the collection of several Banach-space valued paths with certain algebraic structure. We emphasize here an alternative algebraic/geometrical approach, and develop an integration directly for one-forms against group-valued paths. The generalization is needed to get the algebraic closure of integrals. Indeed, suppose w is Brownian motion, and γ^i , $i = 1, 2$, are suitable integrands. We want to find γ that satisfies

$$\int_0^\cdot \gamma_u dw_u \stackrel{?}{=} \int_0^\cdot \gamma_u^1 dw_u \int_0^\cdot \gamma_u^2 dw_u.$$

Based on Itô's lemma there exists a drift term in the product that can not be represented in the form of an integral against w , so such γ does not exist (see Theorem 1 [11] for a pathwise generalization). A generalized integral is therefore needed to get the algebraic closure (even for almost all Brownian sample paths) that is important for proving density in paths space. The integral developed here is rich enough to handle the product structure of rough integrals, and the multiplication is in fact a continuous operation in the space of one-forms (Proposition 35). In particular, suppose y^i , $i = 1, 2$, solves the Itô differential equation $dy^i = f(y^i) dw$, $y_0^i = \xi^i$, for Brownian motion w . By using the integral developed in this paper, y^i and their product $y^1 y^2$ solve the same type of equation, and the product $(y^1, y^2) \mapsto y^1 y^2$ is a continuous operation. The algebra structure is compatible with the filtration generated on paths space, and the product of two previsible integrable one-forms is another previsible integrable one-form. This is the same property that was exploited in the Martingale Representation Theorem.

By introducing a family of closed one-forms on a group, we construct the integral of one-forms against group-valued paths, and provide a simple and unified interpretation of the extension theorem and the theories of integration in rough paths theory [25, 14, 15]. We are able to allow the one-form on the group to vary with time. As a consequence, the integral is not restricted to use the same cotangent vector at distinct times of self-intersection.

We identify structural properties of the group that enable basic operations, e.g. rough integration and iterated integration. Condition 25' is for rough integration, and encodes the information of how to

integrate monomials against degree-one monomial on paths space (Corollary 43, Remark 47). Condition 25 is for iterated integration that encodes the information of how to integrate monomials against monomials (not only degree-one monomial) on paths space (Proposition 32, Corollary 48). These algebraic conditions are structural assumptions on the group, and can be viewed as counterparts to Chen's identity (that is about paths evolution) in paths integration.

The process of rough integration can be split into two (independent) steps: (1) integrating polynomials and increasing the regularity of the Lipschitz function (2) constructing a group homomorphism from the Lipschitz function. The integration we constructed (Theorem 15) is about the homomorphism, and is not (directly) related to increasing the regularity of the Lipschitz function. The regularity of the Lipschitz function is increased when we specify the one-form (Corollary 43, Remark 47).

By using one-forms, basic operations — such as iterated integration, multiplication and composition with regular functions — are continuous operations (Section 4). The continuity gives considerable analytical robustness. In particular, the enhancement to a group-valued path is a continuous operation. That is applicable when the enhancement involves basic operations e.g. taking values in nilpotent Lie group or Butcher group.

The approach is employed in [27] to extend an argument of Schwartz [29] to rough differential equations, and give a short proof of the global unique solvability and stability of the solution that is applicable to geometric rough paths and branched rough paths. Consider the rough differential equation

$$dy = f(y) dx, y_0 = \xi,$$

with Picard iterations $y^n := \xi + \int_0^\cdot f(y^{n-1}) dx$, $n \geq 1$, $y^0 \equiv \xi$. When f is $\text{Lip}(\gamma)$ for $\gamma > p$, there exist one-forms $(\beta^n)_n$ such that $y^n = \xi + \int_0^\cdot \beta^n(g) dg$ and $(\beta^{n+1} - \beta^n)_n$ decay factorially in operator norm (Theorem 22 [27]). More explicitly, there is a constant $C = C(p, \gamma, \|f\|_{\text{Lip}(\gamma)}, \omega(0, T))$ such that, with $\|\cdot\|_\theta^\omega$ in (3.10), control $\omega := \|g\|_{p\text{-var}}^p$ and $\theta := \frac{\gamma \wedge (p+1)}{p} > 1$, we have

$$\|\beta^{n+1} - \beta^n\|_\theta^\omega \leq \frac{C^{n-[p]}}{\left(\frac{n-[p]}{p}\right)!}, n \geq [p] + 1.$$

Since the enhancement to a group-valued path is a continuous operation in the space of one-forms and the indefinite integral is a continuous operation from one-forms to paths, we have the convergence of Picard iterations and their group-valued enhancements.

The basic idea of the integral in this paper can be summarized as follows. Suppose x is a continuous bounded variation path taking values in \mathbb{R}^d and α is a one-form. Let $G^{(n)}(\mathbb{R}^d)$ denote the step- n free nilpotent Lie group over \mathbb{R}^d (see [25]) with projection π onto \mathbb{R}^d . Then we may lift x up to a path g taking values in $G^{(n)}(\mathbb{R}^d)$ (the signature of x), pull α up to α^* using π , and get

$$\int \alpha(x) dx = \int \alpha^*(g) dg.$$

This equality holds because $x = \pi g$ and has little to do with $g_t \in G^{(n)}(\mathbb{R}^d)$. Since the dimension of the tangent space to g in $G^{(n)}(\mathbb{R}^d)$ is much larger than that of \mathbb{R}^d , there are other choices of one-forms that give the same integral. In particular, when $\alpha = p$ is a polynomial one-form of degree $(n-1)$, there is a unique closed (in fact cocyclic) one-form P on $G^{(n)}(\mathbb{R}^d)$ (as defined in Example 3 below) that only depends on p and satisfies

$$\int p(x) dx = \int P(g) dg. \quad (1.1)$$

Because this statement holds exactly rather than infinitesimally, it gives considerable analytical flexibility. As we suggested before, Lipschitz functions are continuous functions taking values in polynomial functions. Since each polynomial one-form can be lifted to a closed one-form on the nilpotent Lie group, a Lipschitz one-form can be lifted to a continuous function taking values in closed one-forms on the nilpotent Lie group. In particular for Lipschitz function α , if we denote the lift of α by β , then for any continuous bounded variation path x with lift g ,

$$\int \alpha(x) dx = \int \beta(g) dg. \quad (1.2)$$

When x is not necessarily of bounded variation, the integrals $\int p(x) dx$ in (1.1) and $\int \alpha(x) dx$ in (1.2) may not be meaningfully defined. Since P is a closed one-form, the integral $\int P(g) dg$ is well defined

for any continuous path g . Since β is a function taking values in closed one-forms in the form of P , the integral $\int \beta(g) dg$ should still make sense when the closed one-form varies slowly. For example, in the extreme case that β is a constant closed one-form P , $\int \beta(g) dg$ coincides with $\int P(g) dg$ so is well defined. Indeed, it can be proved that when β is the lift of a $Lip(\gamma)$ function and g is a continuous path with finite p -variation for $\frac{\gamma+1}{p} > 1$, the integral $\int \beta(g) dg$ is well defined and coincides with the rough integral in [25]. The existence of the integral $\int \beta(g) dg$ relies on that β consists a family of well-behaved one-forms (the P s) that vary slowly along the trajectory of g (captured by the Lipschitz degree). Closed one-forms are well-behaved because they put little assumption on the path for the integral to make sense, and they can serve as basic ingredients in the space of more general one-forms (like constant one-forms in the space of continuous one-forms in the classical integration). Generally, the integral $\int_{t \in [0, T]} \beta_t(g_t) dg_t$ is well defined, when β_t is a continuous path taking values in closed one-forms and the two dual paths β_t and g_t satisfy a generalized Young condition (see construction of the integral in Section 2.2).

Polynomial one-forms are basic ingredients for the rough integration in [25], and serve as the primary example in this paper.

1.2 Cocyclic one-forms

Cocyclic one-forms are closed one-forms on a topological group. They can be integrated against *any* continuous path taking values in the group, and the value of the integral only depends on the path through end points.

Suppose \mathcal{A} and \mathcal{B} are two Banach algebras and \mathcal{G} is a topological group in \mathcal{A} . We denote by $L(\mathcal{A}, \mathcal{B})$ the set of continuous linear mappings from \mathcal{A} to \mathcal{B} , and denote by $C(\mathcal{G}, L(\mathcal{A}, \mathcal{B}))$ the set of continuous mappings from \mathcal{G} to $L(\mathcal{A}, \mathcal{B})$.

Definition 1 (Cocyclic One-Form) *We say $\beta \in C(\mathcal{G}, L(\mathcal{A}, \mathcal{B}))$ is a cocyclic one-form, if there exists a topological group \mathcal{H} in \mathcal{B} such that $\beta(a, b) \in \mathcal{H}$, $\forall a, b \in \mathcal{G}$, and*

$$\beta(a, b) \beta(ab, c) = \beta(a, bc), \forall a, b, c \in \mathcal{G}. \quad (1.3)$$

We denote the set of cocyclic one-forms by $B(\mathcal{G}, \mathcal{H})$ (or $B(\mathcal{G})$).

Equation (1.3) represents the exact equality between the one-step and two-steps estimates that characterizes closed one-forms. Intuitively, if we start from point a and go in the direction of bc , then it is equivalent to start from point a , go in the direction of b , and start again from point ab and then go in the direction of c .

Cocyclic one-forms are of specific form but abundant; they are fundamental in integration. A simple example is the constant one-form on a Banach space, which we use implicitly in the classical integration. Another example is the polynomial one-form in rough paths theory [25].

Recall that a polynomial function of degree (at most) n is a globally defined function whose $(n+1)^{\text{th}}$ derivative exists and is identically zero.

Definition 2 (Polynomial One-Form) *For Banach spaces \mathcal{V} and \mathcal{U} , we say $p \in C(\mathcal{V}, L(\mathcal{V}, \mathcal{U}))$ is a polynomial one-form of degree n if p is a polynomial function of degree n taking values in $L(\mathcal{V}, \mathcal{U})$.*

In particular, by using Taylor's theorem, we have (with \otimes denoting the tensor product)

$$p(z)(v) = \sum_{l=0}^n (D^l p)(y) \frac{(z-y)^{\otimes l}}{l!}(v), \forall z, v, y \in \mathcal{V}, \quad (1.4)$$

where $(D^l p)(y) \in L(\mathcal{V}^{\otimes l}, L(\mathcal{V}, \mathcal{U}))$ denotes the value at y of the l -th derivative of p . Like polynomial functions, there is neither a given point nor a power series associated with a polynomial one-form. One can choose different representations of a polynomial one-form by choosing the point y in (1.4), and the value of p does not depend on the choice of y .

Then we enhance a polynomial one-form to a cocyclic one-form on a group. Let $G^{(n)}(\mathcal{V})$ denote the step- n free nilpotent Lie group over Banach space \mathcal{V} , which is canonically embedded in the Banach algebra $T^{(n)}(\mathcal{V}) = \mathbb{R} \oplus \mathcal{V} \oplus \cdots \oplus \mathcal{V}^{\otimes n}$, and let π_k denote the projection of $T^{(n)}(\mathcal{V})$ to $\mathcal{V}^{\otimes k}$. For integers $l_i \geq 1$, $i = 1, 2, \dots, k$, we let

$$OS(l_1, l_2, \dots, l_k)$$

denote the ordered shuffles of k stacks of cards with l_1, l_2, \dots, l_k cards respectively (p.73, [26]).

Example 3 (Polynomial Cocyclic One-Form) Suppose $p \in C(\mathcal{V}, L(\mathcal{V}, \mathcal{U}))$ is a polynomial one-form of degree $(n-1)$ for some integer $n \geq 1$. We define $P \in C(G^{(n^2)}(\mathcal{V}), L(T^{(n^2)}(\mathcal{V}), T^{(n)}(\mathcal{U}))$ by, for $a \in G^{(n^2)}(\mathcal{V})$ and $v \in T^{(n^2)}(\mathcal{V})$,

$$P(a, v) := 1 + \sum_{k=1}^n \sum_{l_i \in \{0,1,\dots,n-1\}} ((D^{l_1}p) \otimes \dots \otimes (D^{l_k}p)) (\pi_1(a)) \sum_{\rho \in OS(l_1+1, \dots, l_k+1)} \rho^{-1}(\pi_{l_1+\dots+l_k+k}(v)). \quad (1.5)$$

Then P is a cocyclic one-form, i.e. for $a, b \in G^{(n^2)}(\mathcal{V})$, $P(a, b) \in G^{(n)}(\mathcal{U})$, and

$$P(a, b) P(ab, c) = P(a, bc), \quad \forall a, b, c \in G^{(n^2)}(\mathcal{V}).$$

For an explanation of the mathematical expression, we suppose x is a continuous bounded variation path on $[0, T]$ taking values in \mathcal{V} , and let $S_n(x)$ denote the step- n Signature of x :

$$S_n(x)_{s,t} := 1 + \sum_{k=1}^n x_{s,t}^k \text{ with } x_{s,t}^k := \int \dots \int_{s < u_1 < \dots < u_k < t} dx_{u_1} \otimes \dots \otimes dx_{u_k}, \quad \forall 0 \leq s \leq t \leq T. \quad (1.6)$$

Based on Chen [5], $S_n(x)$ takes values in the step- n nilpotent Lie group $G^{(n)}(\mathcal{V})$, and satisfies:

$$(\text{Chen's Identity}) \quad S_n(x)_{s,u} S_n(x)_{u,t} = S_n(x)_{s,t}, \quad \forall 0 \leq s \leq u \leq t \leq T,$$

where the multiplication on the l.h.s. is in $G^{(n)}(\mathcal{V})$. In particular, $[0, T] \ni t \mapsto S_n(x)_{0,t} \in G^{(n)}(\mathcal{V})$ is a group-valued path satisfying $S_n(x)_{0,s}^{-1} S_n(x)_{0,t} = S_n(x)_{s,t}$, $\forall 0 \leq s \leq t \leq T$. Since $(D^l p)(x_s) \in L(\mathcal{V}^{\otimes l}, L(\mathcal{V}, \mathcal{U}))$ is symmetric in $\mathcal{V}^{\otimes l}$ and the projection of $x_{s,t}^l$ to the space of symmetric tensors is $(l!)^{-1} (x_t - x_s)^{\otimes l}$ (see [25]), we have

$$(D^l p)(x_s) \frac{(x_t - x_s)^{\otimes l}}{l!} (v) = (D^l p)(x_s) (x_{s,t}^l) (v), \quad \forall v \in \mathcal{V}, \quad \forall 0 \leq s \leq t \leq T. \quad (1.7)$$

Then, based on the expressions (1.4) and (1.7), and by using $x_{s,t}^{l+1} = \int_s^t x_{s,r}^l \otimes dx_r$, $l = 0, \dots, n-1$, we have

$$\int_s^t p(x_r) dx_r = \sum_{l=0}^{n-1} (D^l p)(x_s) \int_s^t \frac{(x_r - x_s)^{\otimes l}}{l!} \otimes dx_r = \sum_{l=0}^{n-1} (D^l p)(x_s) \int_s^t x_{s,r}^l \otimes dx_r = \sum_{l=0}^{n-1} (D^l p)(x_s) x_{s,t}^{l+1},$$

where $(D^l p)(x_s)$ contracts with $x_{s,t}^{l+1}$ because $(D^l p)(x_s) \in L(\mathcal{V}^{\otimes l}, L(\mathcal{V}, \mathcal{U}))$ so $(D^l p)(x_s) (x_{s,t}^l) \in L(\mathcal{V}, \mathcal{U})$ and $(D^l p)(x_s) \int_s^t x_{s,r}^l \otimes dx_r = (D^l p)(x_s) x_{s,t}^{l+1} \in \mathcal{U}$. Hence, for $0 \leq s \leq u \leq t \leq T$, by using (1.4) and that $\int_u^t = \int_s^t - \int_s^u$, we have

$$\begin{aligned} \sum_{l=0}^{n-1} (D^l p)(x_u) x_{u,t}^{l+1} &= \sum_{l=0}^{n-1} (D^l p)(x_u) \int_u^t x_{u,r}^l \otimes dx_r = \int_u^t p(x_r) dx_r = \sum_{l=0}^{n-1} (D^l p)(x_s) \int_u^t x_{s,r}^l \otimes dx_r \\ &= \sum_{l=0}^{n-1} (D^l p)(x_s) x_{s,t}^{l+1} - \sum_{l=0}^{n-1} (D^l p)(x_s) x_{s,u}^{l+1}. \end{aligned}$$

As a result, if we define path $y : [0, T] \rightarrow \mathcal{U}$ by

$$y_t := \int_0^t p(x_r) dx_r = \sum_{l=0}^{n-1} (D^l p)(x_0) x_{0,t}^{l+1}, \quad \forall 0 \leq t \leq T,$$

then

$$y_t - y_s = \sum_{l=0}^{n-1} (D^l p)(x_s) x_{s,t}^{l+1}, \quad \forall 0 \leq s \leq t \leq T. \quad (1.8)$$

Based on the definition of the Signature in (1.6) and the representation of y in (1.8), we have

$$S_n(y)_{s,t} = 1 + \sum_{k=1}^n \sum_{l_i \in \{0,1,\dots,n-1\}} ((D^{l_1}p) \otimes \dots \otimes (D^{l_k}p))(x_s) \int \dots \int_{s < u_1 < \dots < u_k < t} dx_{s,u_1}^{l_1+1} \otimes \dots \otimes dx_{s,u_k}^{l_k+1}. \quad (1.9)$$

By using the ordered shuffle product (p. 73, [26]), $\int \cdots \int_{s < u_1 < \cdots < u_k < t} dx_{s,u_1}^{l_1+1} \otimes \cdots \otimes dx_{s,u_k}^{l_k+1}$ can be rewritten as a universal continuous linear function of $S_{n^2}(x)_{s,t}$ that is independent of x :

$$\int \cdots \int_{s < u_1 < \cdots < u_k < t} dx_{s,u_1}^{l_1+1} \otimes \cdots \otimes dx_{s,u_k}^{l_k+1} = \sum_{\rho \in OS(l_1+1, \dots, l_k+1)} \rho^{-1} \left(\pi_{l_1+\cdots+l_k+k}(S_{n^2}(x)_{s,t}) \right). \quad (1.10)$$

As we mentioned before, if we denote $g_t^{n^2} := S_{n^2}(x)_{0,t}$, $\forall t \in [0, T]$, then g^{n^2} is a group-valued path taking values in the step- n^2 free nilpotent Lie group $G^{(n^2)}(\mathcal{V})$. Based on the representations (1.9) and (1.10), if we define P as in (1.5), then, with $g_{s,t}^{n^2} := (g_s^{n^2})^{-1} g_t^{n^2}$,

$$P(g_s^{n^2}, g_{s,t}^{n^2}) = S_n(y)_{s,t}, \quad \forall 0 \leq s \leq t \leq T. \quad (1.11)$$

Based on (1.11), P takes values in the step- n nilpotent Lie group $G^{(n)}(\mathcal{U})$ and satisfies Chen's identity:

$$P(g_s^{n^2}, g_{s,u}^{n^2}) P(g_u^{n^2}, g_{u,t}^{n^2}) = P(g_s^{n^2}, g_{s,t}^{n^2}), \quad \forall 0 \leq s \leq u \leq t \leq T. \quad (1.12)$$

Comparing (1.12) with the cocyclic property defined at (1.3), we have that, P is a cocyclic one-form on group $G^{(n^2)}(\mathcal{V})$ taking values in another group $G^{(n)}(\mathcal{U})$.

That polynomial one-forms can be lifted to closed (cocyclic) one-forms is not a coincidence, and follows from properties of polynomials on paths space. A polynomial is a finite linear combination of monomials of an indeterminate. A polynomial on paths space in our setting is a finite linear combination of monomials of a path, i.e. its iterated integrals (see Chen [4] for geometrical interpretation of monomials on paths space). (Following from algebraic properties of the group, classical polynomials correspond naturally to polynomials on paths space, see Corollary 43 and Remark 47.) The space of paths (modulus translations, reparametrisations and tree-like equivalence [18, 1]) forms a group when equipped with the product given by concatenation [25]. The step- n signature $S_n(x)$ can be viewed as the graded composition of the first n monomials of the path x . Thus Chen's identity encodes the change of the representation of a polynomial on paths space as the reference point (a path) changes. In particular, for a continuous linear mapping $A : T^{(n)}(\mathcal{V}) \rightarrow \mathcal{U}$, $p(x) := AS_n(x)$ is the representation of a polynomial when the reference point is the constant path. Based on Chen's identity, the representation of p at a path y is $p(x) = AS_n(y) S_n(\overleftarrow{y} \sqcup x)$ (with $\overleftarrow{y} \sqcup x$ denotes the concatenation of y backwards with x). From this perspective, rough integration (for geometric rough paths and branched rough paths alike) can be viewed as the sewing process of constructing a group homomorphism. Indeed, for a given rough path, rough integration constructs from a family of polynomials (indexed by the evolution of the rough path) a homomorphism (from paths space to another group) with prescribed local behavior (the homomorphism and the Lipschitz function are locally equivalent). That the family of polynomials on paths space is stable under the signature mapping is based on algebraic properties of the group — the existence of the mapping \mathcal{I} in Condition 25. The cocyclic one-form associated with the signature is simple: $\beta(a, b) = b$, $\forall a, b \in \mathcal{G}$, so $\beta(a, b) \beta(ab, c) = bc = \beta(a, bc)$, $\forall a, b, c \in \mathcal{G}$. The equality (1.12) is hence a synthesis of Chen's identity about changing the reference point of a polynomial and the stability of polynomials under the signature mapping, encodes the change of a cocyclic one-form under the change of the base group-valued path, and is a transitive property (see also Proposition 40).

The following diagram illustrates the relationship between paths and one-forms:

$$\begin{array}{ccccc} C([0, T], G^{(n)}(\mathcal{V})) & \ni & g^n & + & \beta & \in & C([0, T], B(G^{(n)}(\mathcal{V}), G^{(n)}(\mathcal{U}))) \\ & & \updownarrow \text{truncation / extension} & & & & \\ C([0, T], G^{(n^2)}(\mathcal{V})) & \ni & g^{n^2} & + & P & \in & B(G^{(n^2)}(\mathcal{V}), G^{(n)}(\mathcal{U})) \\ & & \updownarrow \text{projection / signature} & & & & \\ C([0, T], \mathcal{V}) & \ni & x & + & p & \in & C(\mathcal{V}, L(\mathcal{V}, \mathcal{U})) \end{array}$$

The lower half-diagram summarizes the polynomial one-form and the polynomial cocyclic one-form we discussed above. Suppose x is a continuous bounded variation path on $[0, T]$ taking values in \mathcal{V} and $p \in C(\mathcal{V}, L(\mathcal{V}, \mathcal{U}))$ is a polynomial one-form. For integer $n \geq 1$, we can enhance x to its step- n^2 signature

g^{n^2} , which is a continuous path taking values in the step- n^2 free nilpotent Lie group $G^{(n^2)}(\mathcal{V})$. (We put a dashed arrow because, when x is less regular, e.g. a Brownian sample path, the enhancement exists but may not be unique, see [24].) We can define the polynomial cocyclic one-form P associated with p on $G^{(n^2)}(\mathcal{V})$ taking values in $G^{(n)}(\mathcal{U})$ as in (1.5). Then based on (1.11),

$$P(g_s^{n^2}, g_{s,t}^{n^2}) = S_n \left(\int_0^{\cdot} p(x_u) dx_u \right)_{s,t}, \quad \forall 0 \leq s < t \leq T. \quad (1.13)$$

The upper half-diagram is about truncation and keeping the homogeneity of the degree of the groups so that it would not explode after several compositions. The lifting to degree- n^2 is necessary to keep a closed expression: although polynomials on paths space form an algebra, polynomials of a specific degree do not form one — the degree- n polynomial of a degree- n polynomial is a degree- n^2 polynomial. If we would like to define a one-form on $G^{(n)}(\mathcal{V})$ (instead of $G^{(n^2)}(\mathcal{V})$), we can truncate g^{n^2} to g^n , and define β based on the truncation of P as a time-varying cocyclic one-form on $G^{(n)}(\mathcal{V})$ (see Corollary 43). When g^n is of finite p -variation for some $[p] \leq n$, β is integrable against g^n and satisfies

$$\int_s^t \beta_u(g_u^n) dg_u^n = P(g_s^{n^2}, g_{s,t}^{n^2}), \quad \forall 0 \leq s < t \leq T. \quad (1.14)$$

Due to the truncation of higher leveled terms, the β here is no longer a constant cocyclic one-form as P is, and the equality (1.14) follows from the comparison of local expansions. Actually there is no canonical way of enhancing a polynomial one-form to a cocyclic one-form, and one could as well define the cocyclic one-form obtained after truncation as the polynomial cocyclic one-form (denoted by P'). Then the homogeneity of the group is preserved, and β_t is a continuous path taking values in polynomial cocyclic one-forms in the form of P' . In that case, the equality (1.13) (with $P(g_s^{n^2}, g_{s,t}^{n^2})$ replaced by $P'(g_s^n, g_{s,t}^n)$) holds not exactly but approximately with a small error which will disappear in the process of integration, and the equality $\int_s^t \beta_u(g_u^n) dg_u^n = S_n \left(\int_0^{\cdot} p(x_u) dx_u \right)_{s,t}$ still holds. The arrow from g^n to g^{n^2} is about the extension theorem (Theorem 22). Indeed, if g^n is of finite p -variation for some $[p] \leq n$, then there exists a unique g^{n^2} which extends g^n , and g^{n^2} can be represented as the integral of a slow-varying cocyclic one-form against g^n .

By lifting a polynomial one-form on a Banach space to a polynomial cocyclic one-form on the corresponding nilpotent Lie group, we linearize the one-form by working with a larger affine space. The enlargement of the space is sufficient and necessary for developing robust calculus for paths of low regularity. More importantly, we view polynomial one-forms, which are by no means closed one-forms in the classical sense, as closed one-forms on the lifted group (see (1.13)). The closedness of the one-forms relies on the stability of polynomials on paths space, and ultimately on the algebraic structure of the group. The closedness of the one-forms makes the analysis considerably simpler because the integration of a closed one-form puts little restriction on the path. As in the classical integration, where we implicitly work with slowly-varying constant one-forms, we can develop integration of one-forms on a group by slowly-varying the closed (cocyclic) one-forms on the group. In light of the nice approximating properties of polynomials, by slowly-varying the polynomial one-form and by taking closure of the polynomial one-forms w.r.t. an appropriate norm, we can work with a large family of one-forms (e.g. Lipschitz one-forms as in Corollary 43 and time-varying Lipschitz one-forms as in Remark 44).

1.3 Time-varying cocyclic one-forms

As we suggested before, the integration of time-varying cocyclic one-forms reduces to the integration of closed one-forms together with pasting closed one-forms in a consistent way. Before proceeding to the definition of the integral, we take a look at a little lemma, which shares the same spirit with our group-valued integration.

Definition 4 (Finite Partition) $D = \{t_k\}_{k=0}^n$ is a finite partition of $[0, T]$, if $0 = t_0 < t_1 < \dots < t_n = T$. We denote $|D| := \max_k |t_{k+1} - t_k|$.

Lemma 5 For a differentiable manifold M , we suppose x is a continuous path on $[0, T]$ taking values in M , and α is a path on $[0, T]$ taking values in real-valued closed one-forms on M . If there exists $\theta > 1$ such that

$$\left| \int_u^t (\alpha_s - \alpha_u)(x_r) dx_r \right| \leq |t - s|^\theta, \quad \forall 0 \leq s < u < t \leq T, \quad (1.15)$$

then the integral defined by

$$\int_0^T \alpha_r(x_r) dx_r := \lim_{|D| \rightarrow 0, D=\{t_k\}_{k=0}^n \subset [s, t]} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \alpha_{t_k}(x_r) dx_r, \quad \forall 0 \leq s < t \leq T, \quad (1.16)$$

exists, and satisfies, for some constant $C_\theta > 0$,

$$\left| \int_s^t (\alpha_r - \alpha_s)(x_r) dx_r \right| \leq C_\theta |t - s|^\theta, \quad \forall 0 \leq s < t \leq T. \quad (1.17)$$

For each $u \in [0, T]$, α_u is a closed one-form, so the integral of α_u against any continuous path on M is simple and only depends on end points of the path. The compensated regularity condition (1.15) between paths α and x guarantees that we can change from α_s to α_u at x_u without a big error. Then, as in the case of Young integral [31], we can sequentially remove partition points and the integral exists with the local estimate (1.17). The idea is the same if we try to integrate a slow-varying cocyclic one-form against a group-valued path, because cocyclic one-forms are closed one-forms on a group.

Suppose g is a continuous path on $[0, T]$ taking values in group \mathcal{G} , and β is a time-varying cocyclic one-form (or say, a continuous path taking values in cocyclic one-forms on \mathcal{G}).

Definition 6 (Integral) Let $g \in C([0, T], \mathcal{G})$ and $\beta \in C([0, T], B(\mathcal{G}, \mathcal{H}))$. If the limit exists

$$\lim_{|D| \rightarrow 0, D=\{t_k\}_{k=0}^n \subset [0, T]} \beta_0(g_0, g_{0,t_1}) \beta_{t_1}(g_{t_1}, g_{t_1,t_2}) \cdots \beta_{t_{n-1}}(g_{t_{n-1}}, g_{t_{n-1},T}), \quad \text{with } g_{s,t} := g_s^{-1} g_t, \quad (1.18)$$

then we define the limit to be the integral $\int_0^T \beta_u(g_u) dg_u$.

For each $u \in [0, T]$, β_u is a cocyclic one-form. The integral of β_u against any continuous path g only depends on the end points, and satisfies $\int_s^t \beta_u(g_r) dg_r = \beta_u(g_s, g_{s,t})$, $\forall s < t$. Hence, (1.18) can be rewritten as

$$\lim_{|D| \rightarrow 0, D=\{t_k\}_{k=0}^n \subset [0, T]} \int_0^{t_1} \beta_0(g_u) dg_u \int_{t_1}^{t_2} \beta_{t_1}(g_u) dg_u \cdots \int_{t_{n-1}}^T \beta_{t_{n-1}}(g_u) dg_u,$$

which is similar to (1.16). The proof of the existence of the integral is also similar. We will assume a generalized Young condition (Condition 14), and get a local estimate in the form of (1.17) (Theorem 15). Since cocyclic one-forms take values in another Banach algebra, if we integrate a time-varying cocyclic one-form, we would need some graded structure of the target Banach algebra for the integral to make sense (specified in Section 2.1, see [9] for a general condition for the limit to exist) and these assumptions set out the basis on which dominated paths are defined. In particular, we assume that the Banach algebra coincides with $\mathbb{R} \oplus \mathcal{V} \oplus \cdots \oplus \mathcal{V}^{\otimes n}$ as a Banach space and that the multiplication in the Banach algebra is induced by the comultiplication of a family of graded projective mappings. Since a Banach space can be canonically embedded in a graded Banach algebra, our formulation includes Banach-space valued one-forms as special cases (e.g. dominated paths as in Definition 28). For polynomial cocyclic one-forms, we can vary it with time to incorporate Lipschitz one-forms as in [25] (Corollary 43) and also incorporate time-varying Lipschitz one-forms (Remark 44). In particular, the rough integral $\int_{u \in [s, t]} \alpha(X_u, u) dX_u$ with an inhomogeneous degree of smoothness assumption on α can be treated as a special example of integrating time-varying Lipschitz one-forms (Remark 45).

In proving the existence of the integral and in defining the dominated paths, we would need to compare cocyclic one-forms. Suppose that we want to switch from one one-form to another at a point (say a). Since they are cocyclic, if they are close at a , then they will be close on the whole group pointwisely (by which we mean that if a sequence of cocyclic one-forms converge at a specific point then they converge on the whole group pointwisely). However, if we want to identify a fairly sharp regularity condition on the one-forms to integrate against a given group-valued path, then it is reasonable to compare these two one-forms only around a , because the difference between two one-forms will propagate based on the structure of the group. Take the polynomial one-form as an example. Suppose p and q are two degree $(n-1)$ polynomial one-forms, which are close at 0. Then for $l = 0, 1, \dots, n-1$,

$$(D^l p)(z) - (D^l q)(z) = \sum_{k=0}^{n-1-l} ((D^{l+k} p)(0) - (D^{l+k} q)(0)) \frac{z^{\otimes k}}{k!}, \quad \forall z \in \mathcal{V}. \quad (1.19)$$

For $z \in \mathcal{V}$ satisfying $\delta \leq \|z\| \leq \delta^{-1}$ for some $\delta \in (0, 1)$, based on the expression (1.19), we have the estimate $\|(D^l p)(z) - (D^l q)(z)\| \lesssim \max_{k=0}^{n-1-l} \|(D^{l+k} p)(0) - (D^{l+k} q)(0)\|$ (rather than $\|(D^l p)(z) - (D^l q)(z)\| \lesssim \|(D^l p)(0) - (D^l q)(0)\|$). In the theory of rough paths we use an inhomogeneous distance between two one-forms to compensate the inhomogeneous norm on the group where the paths live. Based on (1.19), this inhomogeneous distance will not be preserved if we compare these two one-forms at a point that is far from our reference point. Hence, we compare these two one-forms around a as two continuous linear operators on a graded Banach-space, equipped with an inhomogeneous norm. In fact, the definition of the norm (homogeneous or inhomogeneous) is not essential in the construction of integral. As long as the two paths — one takes values in one-forms and the other takes values in the group — satisfy a compensated regularity condition (Condition 14), the integral is well-defined and is reduced to an analogue of the classical Young integral.

We introduce cocyclic one-forms as a family of closed one-forms on a topological group, and recast the integration in the theory of rough paths as an example of integrating a time-varying cocyclic one-form against a group-valued path. Under a compensated regularity condition between the one-form and the path, the integral exists, and the indefinite integral obtained is another group-valued path. The integral recovers and extends the theories of integration for geometric rough paths and for branched rough paths [25, 14, 15]. As an application, we prove the extension theorem that there exists a unique extension of a given group-valued path of finite p -variation, and we represent the extended path as the integral of a time-varying cocyclic one-form against the original group-valued path.

For a group-valued path, we define dominated paths as a family of Banach-space valued paths that can be represented as integrals of time-varying cocyclic one-forms against the given group-valued path. Under some structural assumptions on the group, the set of dominated paths is both a linear space and an algebra, has a canonical enhancement to a group-valued path, is stable under composition with regular functions and satisfies a transitive property. There are minor differences between dominated paths and controlled paths as defined in [14, 15]. Some discussion about their relationship can be found in Section 3.3. In particular, dominated paths are defined from and determined by one-forms, and problems about dominated paths can be reformulated in terms of one-forms. Working with one-forms has the benefit that basic operations (iterated integration, multiplication, composition, transitivity) are continuous in the space of one-forms (Section 4). For example, the solution to a rough differential equation can equally be formulated as a fixed point problem in the space of integrable one-forms. The enhancement into a group-valued path is a continuous operation in the space of one-forms (based on (4.2) (4.10)) and integration is a continuous operation from one-forms to paths (based on (3.13)), so if the one-forms associated with a sequence of dominated paths converge then their group-valued enhancements also converge. In [27], we give an accessible overview of the one-form approach developed here. As an application, we extend an argument of Schwartz [29] to rough differential equations, and give a short proof of the global unique existence and continuity of the solution by using one-forms.

2 Construction of the integral

Recall in Definition 1 that $\beta \in C(\mathcal{G}, L(\mathcal{A}, \mathcal{B}))$ is called a cocyclic one-form if there exists a topological group \mathcal{H} in \mathcal{B} such that $\beta(a, b) \in \mathcal{H}$, $\forall a, b \in \mathcal{G}$, and

$$\beta(a, bc) = \beta(a, b) \beta(ab, c), \forall a, b, c \in \mathcal{G}. \quad (2.1)$$

Based on (2.1), we have $\beta(a, 1_{\mathcal{G}}) = 1_{\mathcal{H}}$, $\forall a \in \mathcal{G}$, and $\beta(a, b)^{-1} = \beta(ab, b^{-1})$, $\forall a, b \in \mathcal{G}$. Moreover, the one-form at $a \in \mathcal{G}$ only depends on the one-form at $1_{\mathcal{G}}$ and the structure of the group:

$$\beta(a, b) = \beta(1_{\mathcal{G}}, a)^{-1} \beta(1_{\mathcal{G}}, ab), \forall a, b \in \mathcal{G}.$$

The cocyclic one-form is a purely algebraic object; and the topology is coming in when we want to vary it with time.

Proposition 7 *Let $\beta \in C(\mathcal{G}, L(\mathcal{A}, \mathcal{B}))$. Then β is a cocyclic one-form if and only if there exist a topological group \mathcal{H} in \mathcal{B} and $\alpha \in L(\mathcal{A}, \mathcal{B})$ satisfying $\alpha(\mathcal{G}) \subseteq \mathcal{H}$, such that*

$$\beta(a, b) = \alpha(a)^{-1} \alpha(ab), \forall a, b \in \mathcal{G}.$$

Proof. \Leftarrow is clear. For \Rightarrow , set $\alpha(b) := \beta(1_{\mathcal{G}}, b)$, $\forall b \in \mathcal{G}$. Then $\beta(a, b) = \beta(1_{\mathcal{G}}, a)^{-1} \beta(1_{\mathcal{G}}, ab) = \alpha(a)^{-1} \alpha(ab)$, $\forall a, b \in \mathcal{G}$. ■

Proposition 7 is simple, but is useful for constructing a cocyclic one-form.

Recall the definition of the integral in Definition 6, i.e. for $\beta \in C([0, T], B(\mathcal{G}))$ and $g \in C([0, T], \mathcal{G})$,

$$\int_s^t \beta_u(g_u) dg_u := \lim_{|D| \rightarrow 0, D = \{t_k\}_{k=0}^n \subset [s, t]} \beta_0(g_0, g_{0, t_1}) \beta_{t_1}(g_{t_1}, g_{t_1, t_2}) \cdots \beta_{t_{n-1}}(g_{t_{n-1}}, g_{t_{n-1}, t}), \quad \forall s < t, \quad (2.2)$$

provided the limit exists. When $\beta \in C([0, T], B(\mathcal{G}))$ is a constant cocyclic one-form, i.e. $\beta_t \equiv \beta_0 \in B(\mathcal{G})$, we know how to integrate β against $g \in C([0, T], \mathcal{G})$, because based on the cocyclic property in (2.1) and the definition of integral in (2.2),

$$\int_s^t \beta_u(g_u) dg_u = \int_s^t \beta_0(g_u) dg_u = \beta_0(g_s, g_{s, t}), \quad \forall s < t.$$

Then, when $t \mapsto \beta_t$ varies slowly (to be quantified), the integral of β against g should still exist. In that case, the behavior of β_t will depend on the behavior of g_t . In particular, if $g_t \equiv g_0$, then for any $\beta : [0, T] \rightarrow B(\mathcal{G}, \mathcal{H})$, we have $\int_0^T \beta_u(g_u) dg_u = 1_{\mathcal{H}}$ (based on the definition of integral and using that $\beta_s(a, 1_{\mathcal{G}}) = 1_{\mathcal{H}}, \forall a \in \mathcal{G}, \forall s$). More generally, when β and g have compensated regularities, the integral $\int \beta(g) dg$ exists. We give a sufficient condition in Condition 14.

2.1 Algebraic assumptions

The algebraic structure formulated in this section will be assumed throughout the paper. Our structure is similar to that used in [25, 23, 26, 14, 15].

Following Def 1.25 [26], we equip the tensor powers of \mathcal{V} with admissible norms.

Definition 8 (Admissible Norms) *Let \mathcal{V} be a Banach space. We say the tensor powers of \mathcal{V} are equipped with admissible norms, if the following conditions hold ($\text{Sym}(k)$ denotes the symmetric group of degree k)*

$$\begin{aligned} \|\varrho v\| &= \|v\|, \quad \forall v \in \mathcal{V}^{\otimes k}, \quad \forall \varrho \in \text{Sym}(k), \\ \|u \otimes v\| &= \|u\| \|v\|, \quad \forall u \in \mathcal{V}^{\otimes k}, \quad \forall v \in \mathcal{V}^{\otimes j}. \end{aligned}$$

Notation 9 (Triple $(T^{(n)}(\mathcal{V}), \mathcal{G}_n, \mathcal{P}_n)$)

(1) *We assume that $T^{(n)}(\mathcal{V})$ is a unital associative Banach algebra, which, as a Banach space, coincides with $\mathbb{R} \oplus \mathcal{V} \oplus \cdots \oplus \mathcal{V}^{\otimes n}$ with the norm (π_k denotes the projection to $\mathcal{V}^{\otimes k}$)*

$$\|v\| := \sum_{k=0}^n \|\pi_k(v)\|, \quad \forall v \in T^{(n)}(\mathcal{V}).$$

(2) *The multiplication on $T^{(n)}(\mathcal{V})$ is induced by the comultiplication on a finite set of graded projective mappings:*

$$\mathcal{P}_n = \left\{ \sigma \mid \sigma \in L\left(T^{(n)}(\mathcal{V}), \mathcal{V}^{\otimes |\sigma|}\right), |\sigma| = 0, 1, \dots, n \right\}.$$

More specifically,

(2.a) *If we denote by σ_0 the projection of $T^{(n)}(\mathcal{V})$ to \mathbb{R} , then $\sigma_0 \in \mathcal{P}_n$.*

(2.b) *Each $\sigma \in \mathcal{P}_n$ is a continuous linear mapping satisfying $\sigma \circ \sigma = \sigma$, and $a = \sum_{\sigma \in \mathcal{P}_n} \sigma(a)$ for any $a \in T^{(n)}(\mathcal{V})$.*

(2.c) *Let $\Delta : \mathcal{P}_n \rightarrow \mathcal{P}_n \otimes \mathcal{P}_n$ denote the comultiplication on \mathcal{P}_n . Then*

$$\Delta \sigma_0 = \sigma_0 \otimes \sigma_0.$$

For any $\sigma \in \mathcal{P}_n, |\sigma| \geq 1$, there exist an integer $N(\sigma)$ and $\{\sigma^{j,i} \mid j = 1, 2, i = 1, \dots, N(\sigma)\} \subseteq \mathcal{P}_n, |\sigma^{j,i}| \geq 1, |\sigma^{1,i}| + |\sigma^{2,i}| = |\sigma|, \forall i$, such that

$$\Delta \sigma = \sigma \otimes \sigma_0 + \sigma_0 \otimes \sigma + \sum_{i=1}^{N(\sigma)} \sigma^{1,i} \otimes \sigma^{2,i}. \quad (2.3)$$

When $|\sigma| = 1$, we set $N(\sigma) = 0$ and $\Delta \sigma = \sigma \otimes \sigma_0 + \sigma_0 \otimes \sigma$.

(2.d) *The multiplication on $T^{(n)}(\mathcal{V})$ is induced by the comultiplication on \mathcal{P}_n , i.e. for any $a, b \in T^{(n)}(\mathcal{V})$,*

$$\begin{aligned} \sigma_0(ab) &= \sigma_0(a) \sigma_0(b), \\ \sigma(ab) &= \sigma(a) \sigma_0(b) + \sigma_0(a) \sigma(b) + \sum_{i=1}^{N(\sigma)} \sigma^{1,i}(a) \otimes \sigma^{2,i}(b), \quad \sigma \in \mathcal{P}_n, |\sigma| \geq 1. \end{aligned} \quad (2.4)$$

(3) *\mathcal{G}_n is a closed connected topological group in $T^{(n)}(\mathcal{V})$ satisfying $\sigma_0(a) = 1$ for any $a \in \mathcal{G}_n$.*

We assume that \mathcal{G}_n and $T^{(n)}(\mathcal{V})$ have consistent unit, multiplication and topology, but \mathcal{G}_n may be equipped with a different norm.

Notation 10 (Consistent Triples) We say that $\{(T^{(n)}(\mathcal{V}), \mathcal{G}_n, \mathcal{P}_n)\}_{n=0}^\infty$ is a consistent family, if

- (1) Each $(T^{(n)}(\mathcal{V}), \mathcal{G}_n, \mathcal{P}_n)$ is a triple as in Notation 9.
- (2) For $m \geq n \geq 1$, $\mathcal{P}_n = \{\sigma \mid \sigma \in \mathcal{P}_m, |\sigma| \leq n\}$, and the mapping $1_n : T^{(m)}(\mathcal{V}) \rightarrow T^{(n)}(\mathcal{V})$ defined by $1_n(a) = \sum_{\sigma \in \mathcal{P}_n} \sigma(a)$, $\forall a \in T^{(m)}(\mathcal{V})$, is an algebra homomorphism, and induces a group homomorphism from \mathcal{G}_m to \mathcal{G}_n satisfying $1_n(\mathcal{G}_m) = \mathcal{G}_n$.

With $N(\sigma)$ in (2.3), we denote the integer

$$N(n) := (\#\mathcal{P}_n) \vee (\max_{\sigma \in \mathcal{P}_n} N(\sigma)). \quad (2.5)$$

The nilpotent Lie group and Butcher group are the two main examples in this paper.

Example 11 (Nilpotent Lie Group) Let \mathcal{G}_n be the step- n nilpotent Lie group over Banach space \mathcal{V} . Then \mathcal{P}_n is the set of projective mappings $\{\pi_k\}_{k=0}^n$ with π_k denoting the projection of $T^{(n)}(\mathcal{V})$ to $\mathcal{V}^{\otimes k}$ and $\Delta\pi_k = \sum_{j=0}^k \pi_j \otimes \pi_{k-j}$, $k = 0, 1, \dots, n$. In this case, $a \in \mathcal{G}_n$ if and only if

$$\log_n(a) := \sum_{k=1}^n (-1)^{k+1} \frac{1}{k} (a-1)^k \in \mathcal{V} \oplus [\mathcal{V}, \mathcal{V}] \oplus [\mathcal{V}, [\mathcal{V}, \mathcal{V}]] \oplus \dots \oplus [\mathcal{V}, \dots [\mathcal{V}, \mathcal{V}] \dots],$$

where the multiplication in $(a-1)^k$ is in $T^{(n)}(\mathcal{V})$ and $[\mathcal{V}, \dots [\mathcal{V}, \mathcal{V}] \dots]$ is the subspace of $\mathcal{V}^{\otimes k}$ spanned by $[v_1, \dots, [v_{k-1}, v_k]]$, $v_i \in \mathcal{V}$, $i = 1, \dots, k$, with $[u, v] := u \otimes v - v \otimes u$.

Based on a classical theorem in free Lie algebras (Theorem 3.2 [28]), the nilpotent Lie group can be equivalently characterized by using the shuffle product (as exploited in [26]).

Example 12 (Butcher Group) Let \mathcal{G}_n be the Butcher group over \mathbb{R}^d . Then $\mathcal{P}_n = \{\sigma \mid |\sigma| \leq n\}$ is the set of labelled forests of degree less or equal to n , and Δ is the comultiplication in Connes-Kreimer Hopf algebra. In particular, for a labelled tree τ ,

$$\Delta\tau = 1 \otimes \tau + \tau \otimes 1 + \sum_c P^c(\tau) \otimes R^c(\tau),$$

where the sum is over all non-trivial admissible cuts of τ . For a labelled forest $\tau_1 \tau_2 \dots \tau_n$, where τ_i are labelled trees, we have

$$\Delta(\tau_1 \tau_2 \dots \tau_n) = \Delta\tau_1 \Delta\tau_2 \dots \Delta\tau_n.$$

In this case, $a \in \mathcal{G}_n$ if and only if

$$(\sigma_1 \sigma_2)(a) = \sigma_1(a) \sigma_2(a), \quad \forall \sigma_i \in \mathcal{P}_n, \quad |\sigma_1| + |\sigma_2| \leq n. \quad (2.6)$$

For more detailed explanations and concrete examples, please refer to [6] and [15].

When \mathcal{G}_n is the Butcher group, for $a \in \mathcal{G}_n$ and $\sigma \in \mathcal{P}_n$, instead of $\sigma(a) \in (\mathbb{R}^d)^{\otimes |\sigma|}$, we assume $\sigma(a) \in \mathbb{R}$, $\sigma(a) e_\sigma \in (\mathbb{R}^d)^{\otimes |\sigma|}$ and $a = \sum_{\sigma \in \mathcal{P}_n} \sigma(a) e_\sigma$ (with e_σ denotes the tensor coordinate corresponding to σ). In this case, all the components in $\sigma(ab) = \sigma(a) \sigma_0(b) + \sigma_0(a) \sigma(b) + \sum_{i=1}^{N(\sigma)} \sigma^{1,i}(a) \sigma^{2,i}(b)$ and $(\sigma_1 \sigma_2)(a) = \sigma_1(a) \sigma_2(a)$ are real numbers.

2.2 Existence of the integral

Definition 13 (Control) The function $\omega : 0 \leq s \leq t \leq T \rightarrow \overline{\mathbb{R}}^+$ is called a control if ω is continuous, vanishes on the diagonal $\{(s, t) \mid 0 \leq s = t \leq T\}$ and is super-additive: $\omega(s, u) + \omega(u, t) \leq \omega(s, t)$, $\forall 0 \leq s \leq u \leq t \leq T$.

Recall the triple $(T^{(n)}(\mathcal{V}), \mathcal{G}_n, \mathcal{P}_n)$ in Notation 9 with the structural constant $N(n)$ defined in (2.5). Assume \mathcal{B} is a Banach algebra and \mathcal{H} is a topological group in \mathcal{B} . (We do not assume that \mathcal{B} and \mathcal{H} satisfy the conditions in Section 2.1.) Recall the notation of cocyclic one-forms $B(\mathcal{H}, \mathcal{G}_n)$ as in Definition 1. Based on Definition 6, the integral of $\beta \in C([0, T], B(\mathcal{H}, \mathcal{G}_n))$ against $g \in C([0, T], \mathcal{H})$ is defined by

$$\int_0^T \beta_u(g_u) dg_u := \lim_{|D| \rightarrow 0, D = \{t_k\}_{k=0}^n \subset [0, T]} \beta_0(g_0, g_{0, t_1}) \beta_{t_1}(g_{t_1}, g_{t_1, t_2}) \dots \beta_{t_{n-1}}(g_{t_{n-1}}, g_{t_{n-1}, T}),$$

provided the limit exists.

Condition 14 (Integrable Condition) Let $g \in C([0, T], \mathcal{H})$ and $\beta \in C([0, T], B(\mathcal{H}, \mathcal{G}_n))$. Then g and β are said to satisfy the integrable condition, if there exist $M > 0$, control ω and $\theta > 1$ such that

$$\max_{\sigma \in \mathcal{P}_n} \sup_{0 \leq s < t \leq T} \|\sigma(\beta_s(g_s, g_{s,t}))\| \leq M, \quad (2.7)$$

$$\max_{\sigma \in \mathcal{P}_n} \|\sigma((\beta_u - \beta_s)(g_u, g_{u,t}))\| \leq \omega(s, t)^\theta, \quad \forall 0 \leq s < u < t \leq T. \quad (2.8)$$

The proof of Theorem 15 is in the spirit of Young [31] and Lyons [25]. In [9] the authors proved the existence of a unique multiplicative function for every almost multiplicative function taking values in an associative monoid using dyadic partitions, giving an estimate that is a generalization of (2.9).

Theorem 15 (Existence of Integral) Suppose $g \in C([0, T], \mathcal{H})$ and $\beta \in C([0, T], B(\mathcal{H}, \mathcal{G}_n))$ satisfy Condition 14. Then $\int_0^\cdot \beta_u(g_u) dg_u \in C([0, T], \mathcal{G}_n)$ exists, and there exists a constant $C_{n, \theta, M, \omega(0, T)}$, such that

$$\max_{\sigma \in \mathcal{P}_n} \left\| \sigma \left(\int_s^t \beta_u(g_u) dg_u \right) - \sigma(\beta_s(g_s, g_{s,t})) \right\| \leq C_{n, \theta, M, \omega(0, T)} \omega(s, t)^\theta, \quad \forall 0 \leq s < t \leq T. \quad (2.9)$$

Remark 16 Condition (2.8) is a generalized Young's condition [31], representing the compensated regularity between β and g .

Remark 17 The integral $\int \beta(g) dg$ is continuous in the norm

$$\max_{\sigma \in \mathcal{P}_n} \sup_{0 \leq s < t \leq T} \|\sigma(\beta_s(g_s, g_{s,t}))\| + \max_{\sigma \in \mathcal{P}_n} \sup_{0 \leq s < u < t \leq T} \frac{\|\sigma((\beta_u - \beta_s)(g_u, g_{u,t}))\|}{\omega(s, t)^\theta}.$$

Proof. We first prove a uniform bound over all finite partitions. Then we prove the limit exists.

Since $\beta \in C([0, T], B(\mathcal{H}, \mathcal{G}_n))$ and $\sigma_0(\mathcal{G}_n) = 1$, we have

$$\sigma_0(\beta_s(a, b)) = 1, \quad \forall a, b \in \mathcal{H}, \quad \forall s \in [0, T]. \quad (2.10)$$

For $[s, t] \subseteq [0, T]$ and finite partition $D = \{t_j\}_{j=0}^l$ of $[s, t]$, i.e. $s = t_0 < t_1 < \dots < t_l = t$, we denote

$$\beta_{t_{j_1}, t_{j_2}}^D := \beta_{t_{j_1}}(g_{t_{j_1}}, g_{t_{j_1}, t_{j_1+1}}) \cdots \beta_{t_{j_2-1}}(g_{t_{j_2-1}}, g_{t_{j_2-1}, t_{j_2}}), \quad \forall 0 \leq j_1 \leq j_2 \leq l.$$

By using mathematical induction, we first prove that

$$\max_{\sigma \in \mathcal{P}_n} \sup_{D, D \subset [s, t]} \left\| \sigma(\beta_{s,t}^D) - \sigma(\beta_s(g_s, g_{s,t})) \right\| \leq C_{n, \theta, M, \omega(0, T)} \omega(s, t)^\theta, \quad \forall 0 \leq s < t \leq T. \quad (2.11)$$

Using (2.10), (2.11) holds for σ_0 . Suppose (2.11) holds for $\{\sigma \mid |\sigma| \leq k, \sigma \in \mathcal{P}_n\}$ for some $k = 0, 1, \dots, n-1$. Then, using (2.7), we have

$$M_k := \max_{|\sigma| \leq k, \sigma \in \mathcal{P}_n} \sup_{0 \leq s < t \leq T} \sup_{D, D \subset [s, t]} \left\| \sigma(\beta_{s,t}^D) \right\| \leq C_{n, \theta, M, \omega(0, T)}. \quad (2.12)$$

Fix $\sigma \in \mathcal{P}_n$, $|\sigma| = k+1$, $[s, t] \subseteq [0, T]$ and a finite partition $D = \{t_j\}_{j=0}^l \subset [s, t]$. By using that $\beta_{t_{j-1}}$ is a cocyclic one-form, we have

$$\beta_{s,t}^D - \beta_{s,t}^{D \setminus \{t_j\}} = \beta_{s,t_j}^D (\beta_{t_j} - \beta_{t_{j-1}}) (g_{t_j}, g_{t_j, t_{j+1}}) \beta_{t_{j+1}, t}^D.$$

The multiplication on \mathcal{G}_n is induced by the comultiplication on \mathcal{P}_n . Then with I_d denoting the identity function on \mathcal{P}_n , if

$$((\Delta \otimes I_d) \circ \Delta) \sigma = \sum_i \sigma^{1,i} \otimes \sigma^{2,i} \otimes \sigma^{3,i}, \quad (2.13)$$

then

$$\sigma(\beta_{s,t}^D) - \sigma(\beta_{s,t}^{D \setminus \{t_j\}}) = \sum_{i, |\sigma^{2,i}| \geq 1} \sigma^{1,i}(\beta_{s,t_j}^D) \otimes \sigma^{2,i}((\beta_{t_j} - \beta_{t_{j-1}})(g_{t_j}, g_{t_j, t_{j+1}})) \otimes \sigma^{3,i}(\beta_{t_{j+1}, t}^D),$$

where $|\sigma^{2,i}| \geq 1$ because $\sigma_0((\beta_{t_j} - \beta_{t_{j-1}})(g_{t_j}, g_{t_j, t_{j+1}})) = 0$. Since $|\sigma^{1,i}| + |\sigma^{2,i}| + |\sigma^{3,i}| = |\sigma|$, $\forall i$, and $|\sigma^{2,i}| \geq 1$, we have $|\sigma^{1,i}| \vee |\sigma^{3,i}| \leq |\sigma| - 1 = k$. Using the definition of $N(n)$ in (2.5) and the inductive hypothesis (2.12), we have, for $|\sigma| = k + 1$,

$$\begin{aligned} \left\| \sigma \left(\beta_{s,t}^D \right) - \sigma \left(\beta_{s,t}^{D \setminus \{t_j\}} \right) \right\| &\leq C_n M_k^2 \max_{|\sigma'| \leq k+1, \sigma' \in \mathcal{P}_n} \left\| \sigma' \left((\beta_{t_j} - \beta_{t_{j-1}})(g_{t_j}, g_{t_j, t_{j+1}}) \right) \right\| \\ &\leq C_{n, \theta, M, \omega(0, T)} \max_{|\sigma'| \leq k+1, \sigma' \in \mathcal{P}_n} \left\| \sigma' \left((\beta_{t_j} - \beta_{t_{j-1}})(g_{t_j}, g_{t_j, t_{j+1}}) \right) \right\|. \end{aligned}$$

Then, combined with (2.8), we have

$$\left\| \sigma \left(\beta_{s,t}^D \right) - \sigma \left(\beta_{s,t}^{D \setminus \{t_j\}} \right) \right\| \leq C_{n, \theta, M, \omega(0, T)} \omega(t_{j-1}, t_{j+1})^\theta.$$

For finite partition $D = \{t_j\}_{k=0}^l$ of $[s, t]$, (as in Theorem 1.16 [26]) we select an interval $[t_{j-1}, t_{j+1}]$ that satisfies

$$\omega(t_{j-1}, t_{j+1}) \leq \frac{2}{l-1} \omega(s, t). \quad (2.14)$$

By recursively removing partition point t_j that satisfies (2.14) (removing the middle point when $l = 2$), we have

$$\left\| \sigma \left(\beta_{s,t}^D \right) - \sigma \left(\beta_s(g_s, g_{s,t}) \right) \right\| \leq 2^\theta \zeta(\theta) C_{n, \theta, M, \omega(0, T)} \omega(s, t)^\theta = C_{n, \theta, M, \omega(0, T)} \omega(s, t)^\theta.$$

Hence, (2.11) holds for $\sigma \in \mathcal{P}_n$, $|\sigma| = k + 1$, and the induction is complete.

As a consequence, we have

$$M_n := \max_{\sigma \in \mathcal{P}_n} \sup_{0 \leq s < t \leq T} \sup_{D \subset [s, t]} \left\| \sigma \left(\beta_{s,t}^D \right) \right\| \leq C_{n, \theta, M, \omega(0, T)}. \quad (2.15)$$

Then we prove the existence of $\lim_{|D| \rightarrow 0, D \subset [s, t]} \beta_{s,t}^D$. If the limit exists, then (2.9) holds based on (2.11). Suppose $D' = \{s_w\} \subset [s, t]$ is a refinement of $D = \{t_j\}_{j=0}^l \subset [s, t]$ i.e. for $j = 0, 1, \dots, l$, there exists w_j such that $s_{w_j} = t_j$. Similar as above, for $\sigma \in \mathcal{P}_n$, using the linearity of σ and the comultiplication of σ as in (2.13), we have

$$\begin{aligned} \sigma \left(\beta_{s,t}^D \right) - \sigma \left(\beta_{s,t}^{D'} \right) &= \sum_{j=0}^{l-1} \sigma \left(\beta_{s,t_j}^D \left(\beta_{t_j, t_{j+1}}^D - \beta_{t_j, t_{j+1}}^{D'} \right) \beta_{t_{j+1}, t}^{D'} \right) \\ &= \sum_{j=0}^{l-1} \sum_{i, |\sigma^{2,i}| \geq 1} \sigma^{1,i} \left(\beta_{s,t_j}^D \right) \sigma^{2,i} \left(\beta_{t_j, t_{j+1}}^D - \beta_{t_j, t_{j+1}}^{D'} \right) \sigma^{3,i} \left(\beta_{t_{j+1}, t}^{D'} \right). \end{aligned}$$

Then, using the definition of $N(n)$ in (2.5) and the definition of M_n in (2.15), together with the estimate in (2.11), we have (since $\beta_{t_j, t_{j+1}}^D = \beta_{t_j}(g_{t_j}, g_{t_j, t_{j+1}})$, $\forall j$)

$$\begin{aligned} \left\| \sigma \left(\beta_{s,t}^D \right) - \sigma \left(\beta_{s,t}^{D'} \right) \right\| &\leq C_n M_n^2 \sum_{j=0}^{l-1} \max_{\sigma' \in \mathcal{P}_n} \left\| \sigma' \left(\beta_{t_j}(g_{t_j}, g_{t_j, t_{j+1}}) \right) - \sigma' \left(\beta_{t_j, t_{j+1}}^{D'} \right) \right\| \\ &\leq C_{n, \theta, M, \omega(0, T)} \sum_{j=0}^{l-1} \omega(t_j, t_{j+1})^\theta \leq C_{n, \theta, M, \omega(0, T)} \sup_{|v-u| \leq |D|} \omega(u, v)^{\theta-1}. \end{aligned}$$

For two general partitions D and \tilde{D} of $[s, t]$, we have

$$\begin{aligned} \left\| \sigma \left(\beta_{s,t}^D \right) - \sigma \left(\beta_{s,t}^{\tilde{D}} \right) \right\| &\leq \left\| \sigma \left(\beta_{s,t}^D \right) - \sigma \left(\beta_{s,t}^{D \cup \tilde{D}} \right) \right\| + \left\| \sigma \left(\beta_{s,t}^{\tilde{D}} \right) - \sigma \left(\beta_{s,t}^{D \cup \tilde{D}} \right) \right\| \\ &\leq C_{n, \theta, M, \omega(0, T)} \sup_{|v-u| \leq (|D| \vee |\tilde{D}|)} \omega(u, v)^{\theta-1}. \end{aligned}$$

Since $\theta > 1$, we have that

$$\int_s^t \beta_u(g_u) dg_u := \lim_{|D| \rightarrow 0, D \subset [s, t]} \beta_{s,t}^D \text{ exists.}$$

■

2.3 Extension theorem

As an application of the integration constructed in Section 2.2, we prove the extension theorem (as in Theorem 2.2.1 [25], Proposition 9 [14] and Theorem 7.3 [15]). We represent the extended group-valued path as the integral of a time-varying cocyclic one-form against the original group-value path.

Recall the consistent family of triples $\{(T^{(n)}(\mathcal{V}), \mathcal{G}_n, \mathcal{P}_n)\}_{n=0}^\infty$ in Notation 10 and that $\sigma_0 \in \mathcal{P}_n$ denotes the projection of $T^{(n)}(\mathcal{V})$ to \mathbb{R} .

Notation 18 (Group \mathcal{T}_n) For integer $n \geq 0$, we denote by \mathcal{T}_n the closed topological group in $T^{(n)}(\mathcal{V})$ consisting all $a \in T^{(n)}(\mathcal{V})$ satisfying $\sigma_0(a) = 1$.

For $a \in \mathcal{T}_n$, we define a^{-1} as the algebraic polynomial $1 + \sum_{k=1}^n (-1)^k (a - 1)^k$, where the multiplication in $(a - 1)^k$ is in the algebra $T^{(n)}(\mathcal{V})$. Since we assumed that \mathcal{G}_n is closed and $\sigma_0(\mathcal{G}_n) = 1$, \mathcal{G}_n is a closed subgroup of \mathcal{T}_n .

Notation 19 ($\|g\|_{p\text{-var}, [0, T]}$) We equip \mathcal{T}_n (so \mathcal{G}_n) with the norm

$$|a| := \sum_{\sigma \in \mathcal{P}_n, |\sigma| \geq 1} \|\sigma(a)\|^{\frac{1}{|\sigma|}}, \quad \forall a \in \mathcal{T}_n. \quad (2.16)$$

For $p \geq 1$ and $g \in C([0, T], \mathcal{T}_n)$, we define the p -variation of g by

$$\|g\|_{p\text{-var}, [0, T]} := \sup_{D, D \subset [0, T]} \left(\sum_{k, t_k \in D} |g_{t_k, t_{k+1}}|^p \right)^{\frac{1}{p}}, \quad g_{s, t} := g_s^{-1} g_t.$$

Notation 20 ($C^{p\text{-var}}([0, T], \mathcal{T}_n)$) We denote the set of continuous paths of finite p -variation on $[0, T]$ taking values in \mathcal{T}_n by $C^{p\text{-var}}([0, T], \mathcal{T}_n)$ (similarly denote $C^{p\text{-var}}([0, T], \mathcal{G}_n)$).

Let $m \geq n \geq 1$ be integers. We recall the algebra homomorphism 1_n from $T^{(m)}(\mathcal{V})$ to $T^{(n)}(\mathcal{V})$ in Notation 10 defined by $1_n(a) = \sum_{\sigma \in \mathcal{P}_n} \sigma(a)$, $\forall a \in T^{(m)}(\mathcal{V})$. The algebra homomorphism 1_n induces a group homomorphism from \mathcal{G}_m to \mathcal{G}_n that satisfies $1_n(\mathcal{G}_m) = \mathcal{G}_n$. For $p \geq 1$, $[p]$ denotes the largest integer that is less or equal to p .

Let $g \in C^{p\text{-var}}([0, T], \mathcal{G}_{[p]})$ and integer $n \geq [p] + 1$. We prove that the step- n extension of g exists uniquely and can be represented as the integral of a slow-varying cocyclic one-form against g . In general the extended path lives in \mathcal{T}_n but not in \mathcal{G}_n . To guarantee that the extended path takes values in \mathcal{G}_n , we further assume that \mathcal{G}_n is “large” enough to accommodate the extended path. More specifically, we assume that

Condition 21 For any integer $n \geq 1$, there exists a constant $C_n > 0$ such that, for any $a \in \mathcal{G}_n$ there exists $\tilde{a} \in \mathcal{G}_{n+1}$ satisfying $1_n(\tilde{a}) = a$ and $|\tilde{a}| \leq C_n |a|$ (with $|\cdot|$ defined in (2.16)).

Suppose $a \in \mathcal{G}_n$. When \mathcal{G}_n is the step- n nilpotent Lie group, we can let $\tilde{a} := \exp_{n+1}(\log_n a)$ with \log and \exp defined by algebraic series and the lower index n indicates the level of truncation. When \mathcal{G}_n is the step- n Butcher group with $\mathcal{P}'_n = \{\tau \mid |\tau| \leq n\}$ denoting the set of labelled trees of degree less or equal to n , we can let $\tilde{a} := a + \sum_{k=2}^{n+1} \sum_{\tau_i \in \mathcal{P}'_n, |\tau_1| + \dots + |\tau_k| = n+1} \tau_1(a) \cdots \tau_k(a) e_{\tau_1} \otimes \cdots \otimes e_{\tau_k}$ with e_{τ_i} denoting the tensor coordinate corresponding to τ_i . Then it can be checked that, Condition 21 holds in both cases.

Theorem 22 (Extension) Let $p \geq 1$ be a real number. For $g \in C^{p\text{-var}}([0, T], \mathcal{T}_{[p]})$ and integer $n \geq [p] + 1$, there exists a unique $g^n \in C^{p\text{-var}}([0, T], \mathcal{T}_n)$ satisfying $g_0^n = 1 \in \mathcal{T}_n$ and $1_{[p]}(g_t^n) = g_{0, t}$, $\forall t \in [0, T]$. Moreover, there exists $\beta \in C([0, T], B(\mathcal{T}_{[p]}, \mathcal{T}_n))$ such that β and g satisfy the integrable condition (Condition 14) and

$$g_t^n = \int_0^t \beta_u(g_u) dg_u, \quad \forall t \in [0, T].$$

There exists a constant $C_{n, p}$ (which only depends on n and p) such that

$$\|g^n\|_{p\text{-var}, [s, t]} \leq C_{n, p} \|g\|_{p\text{-var}, [s, t]}, \quad \forall 0 \leq s \leq t \leq T. \quad (2.17)$$

If we further assume that g takes values in $\mathcal{G}_{[p]}$ and Condition 21 holds, then g^n takes value in \mathcal{G}_n .

Proof. Uniqueness. Suppose h^1 and h^2 are two extensions of g in \mathcal{T}_n with finite p -variation. Then for $\sigma \in \mathcal{P}_{[p]}$, $\sigma(h^1) = \sigma(h^2)$. For $\sigma \in \mathcal{P}_n \setminus \mathcal{P}_{[p]}$, if suppose $((\Delta \otimes I_d) \circ \Delta) \sigma = \sum_i \sigma^{1,i} \otimes \sigma^{2,i} \otimes \sigma^{3,i}$, then

$$\begin{aligned} & \|\sigma(h_{s,t}^1) - \sigma(h_{s,t}^2)\| \\ & \leq \lim_{|D| \rightarrow 0, D \subset [s,t]} \sum_{k, t_k \in D} \sum_{i, |\sigma^{2,i}| \geq [p]+1} \|\sigma^{1,i}(h_{s,t_k}^1)\| \left\| \sigma^{2,i}(h_{t_k, t_{k+1}}^1) - \sigma^{2,i}(h_{t_k, t_{k+1}}^2) \right\| \|\sigma^{3,i}(h_{t_{k+1}, t}^2)\| \\ & \leq C_{n,p, \|h^1\|_{p-var, [0, T]}, \|h^2\|_{p-var, [0, T]}} \lim_{|D| \rightarrow 0, D \subset [s,t]} \sum_{k, t_k \in D} \left(\|h^1\|_{p-var, [t_k, t_{k+1}]}^{[p]+1} + \|h^2\|_{p-var, [t_k, t_{k+1}]}^{[p]+1} \right) = 0. \end{aligned} \quad (2.18)$$

Existence. We prove by mathematical induction. Denote $g^{[p]} := g$. For $m = [p], \dots, n-1$, we assume $g^m \in C^{p-var}([0, T], \mathcal{T}_m)$, which holds when $m = [p]$, and define $\beta^m \in C([0, T], B(\mathcal{T}_m, \mathcal{T}_{m+1}))$ by

$$\beta_s^m(a, b) := \left(1_m \left((g_s^m)^{-1} a\right)\right)^{-1} 1_m \left((g_s^m)^{-1} ab\right), \forall a, b \in \mathcal{T}_m, \forall s \in [0, T],$$

where we used the implicit identification of \mathcal{T}_m as a subset of \mathcal{T}_{m+1} , and all operations are in \mathcal{T}_{m+1} except the multiplication between a and b is in \mathcal{T}_m . (That β_s^m is a cocyclic one-form follows from Proposition 7.) For $s < u < t$, we have

$$\begin{aligned} (\beta_u^m - \beta_s^m)(g_u^m, g_{u,t}^m) &= 1_m(g_{u,t}^m) - (1_m(g_{s,u}^m))^{-1} 1_m(g_{s,t}^m) \\ &= (1_m(g_{s,u}^m))^{-1} (1_m(g_{s,u}^m) 1_m(g_{u,t}^m) - 1_m(g_{s,t}^m)) \\ &= (1_m(g_{s,u}^m))^{-1} \left(\sum_{\sigma_i \in \mathcal{P}_{[p]}, |\sigma_1| + |\sigma_2| = m+1} \sigma_1(g_{s,u}^m) \sigma_2(g_{u,t}^m) \right). \end{aligned}$$

We assumed that g^m is of finite p -variation. Then since $\beta_s^m(g_s^m, g_{s,t}^m) = 1_m(g_{s,t}^m)$, we have

$$\begin{aligned} & \max_{\sigma \in \mathcal{P}_{m+1}} \sup_{0 \leq s < t \leq T} \|\sigma(\beta_s^m(g_s^m, g_{s,t}^m))\| \leq 1 \vee \|g^m\|_{p-var, [0, T]}^m, \\ & \max_{\sigma \in \mathcal{P}_{m+1}} \|\sigma((\beta_u^m - \beta_s^m)(g_u^m, g_{u,t}^m))\| \leq C_{m+1, \|g^m\|_{p-var, [0, T]}} \|g^m\|_{p-var, [s, t]}^{m+1}, \forall s < t. \end{aligned}$$

Since $m+1 \geq [p]+1 > p$, (β^m, g^m) satisfies the integrable condition. Then by using that $\sigma(\beta_s^m(g_s^m, g_{s,t}^m))$ equals $\sigma(\int_s^t \beta_u^m(g_u^m) dg_u^m)$ when $\sigma \in \mathcal{P}_m$ and $\sigma(\beta_s^m(g_s^m, g_{s,t}^m))$ equals zero when $\sigma \in \mathcal{P}_{m+1} \setminus \mathcal{P}_m$, and combined with the estimate of the integral in Theorem 15, we have

$$\begin{aligned} \max_{\sigma \in \mathcal{P}_{m+1} \setminus \mathcal{P}_m} \left\| \sigma \left(\int_s^t \beta_u^m(g_u^m) dg_u^m \right) \right\| &= \max_{\sigma \in \mathcal{P}_{m+1}} \left\| \sigma \left(\int_s^t \beta_u^m(g_u^m) dg_u^m \right) - \sigma(\beta_s^m(g_s^m, g_{s,t}^m)) \right\| \\ &\leq C_{m+1, p, \|g^m\|_{p-var, [0, T]}} \|g^m\|_{p-var, [s, t]}^{m+1}. \end{aligned}$$

As a result, if we define

$$g_t^{m+1} := \int_0^t \beta_u^m(g_u^m) dg_u^m, \quad t \in [0, T],$$

then

$$\|g^{m+1}\|_{p-var, [0, T]} \leq C_{m+1, p, \|g^m\|_{p-var, [0, T]}} \|g^m\|_{p-var, [0, T]}$$

which holds inductively for $m = [p], \dots, n-1$. Since the constant could be chosen to be monotone in $m+1$ and $\|g^m\|_{p-var, [0, T]}$, we have (with $g^{[p]} := g$)

$$\|g^n\|_{p-var, [0, T]} \leq C_{n, p, \|g\|_{p-var, [0, T]}} \|g\|_{p-var, [0, T]} < \infty. \quad (2.19)$$

Since $1_{[p]}(g^n) = 1_{[p]}(g^{n-1}) = \dots = g$, g^n is an extension of g in \mathcal{T}_n . Combined with the uniqueness, g^n is the unique step- n extension of g with finite p -variation.

Representation. Define $\beta \in C([0, T], B(\mathcal{T}_{[p]}, \mathcal{T}_n))$ by

$$\beta_s(a, b) := (1_{[p]}(g_s^{-1}a))^{-1} 1_{[p]}(g_s^{-1}ab), \quad \forall a, b \in \mathcal{T}_{[p]}, \forall s \in [0, T].$$

As in the case of β^m and g^m , β is integrable against g , and the integral satisfies

$$\max_{\sigma \in \mathcal{P}_n} \left\| \sigma \left(\int_s^t \beta_u(g_u) dg_u \right) - \sigma(g_{s,t}) \right\| \leq C_{n, p, \|g\|_{p-var, [0, T]}} \|g\|_{p-var, [s, t]}^{[p]+1}, \quad \forall 0 \leq s \leq t \leq T. \quad (2.20)$$

By using $1_{[p]} \left(\int \beta(g) dg \right) = 1_{[p]}(g^n)$, using (2.20) and by following similar argument as in (2.18), we have

$$g_t^n = \int_0^t \beta_u(g_u) dg_u, \quad \forall t \in [0, T].$$

The constant in (2.19) can be chosen to be independent of $\|g\|_{p-var, [0, T]}$. For $c > 0$, denote by δ_c the dilation operator i.e. $\delta_c a = \sum_{\sigma} c^{|\sigma|} \sigma(a)$. Without loss of generality, we assume $\|g\|_{p-var, [0, T]} > 0$ and denote $c := \|g\|_{p-var, [0, T]}^{-1}$. Then $\|\delta_c g\|_{p-var, [0, T]} = 1$, and for any $s < t$,

$$\begin{aligned} c \left\| \int_0^t \beta_u(g_u) dg_u \right\|_{p-var, [s, t]} &= \left\| \delta_c \left(\int_0^t \beta_u(g_u) dg_u \right) \right\|_{p-var, [s, t]} = \left\| \int_0^t \beta_u((\delta_c g)_u) d(\delta_c g)_u \right\|_{p-var, [s, t]} \\ &\leq C_{p, n} \|\delta_c g\|_{p-var, [s, t]} = c C_{p, n} \|g\|_{p-var, [s, t]}, \end{aligned}$$

where we used the uniqueness of extension because $1_{[p]}(\delta_c(\int \beta(g) dg)) = \delta_c g = 1_{[p]}(\int \beta(\delta_c g) d\delta_c g)$.

Then we check that, when $g \in C^{p-var}([0, T], \mathcal{G}_{[p]})$ and Condition 21 holds, $\int \beta(g) dg = g^n$ takes values in \mathcal{G}_n . For $m = [p], \dots, n-1$, suppose that g^m takes values in \mathcal{G}_m , which holds when $m = [p]$. Based on Condition 21, there exists a constant $C_m > 0$ such that for any $0 \leq s < t \leq T$ there exists $a_{m+1}^{s, t} \in \mathcal{G}_{m+1}$ such that

$$1_m(a_{m+1}^{s, t}) = g_{s, t}^m \text{ and } \sum_{\sigma \in \mathcal{P}_{m+1} \setminus \mathcal{P}_m} \|\sigma(a_{m+1}^{s, t})\| \leq C_m \|g^m\|_{p-var, [s, t]}^{m+1}.$$

Then we have (with multiplications in \mathcal{T}_{m+1})

$$\begin{aligned} &\|1_m(g_{t_0, t_1}^m) \cdots 1_m(g_{t_{l-1}, t_l}^m) - a_{m+1}^{t_0, t_1} a_{m+1}^{t_1, t_2} \cdots a_{m+1}^{t_{l-1}, t_l}\| \\ &= \left\| \sum_{j=0}^{l-1} \sum_{\sigma \in \mathcal{P}_{m+1} \setminus \mathcal{P}_m} \sigma(a_{m+1}^{t_j, t_{j+1}}) \right\| \leq C_m \sum_{j=0}^{l-1} \|g^m\|_{p-var, [t_j, t_{j+1}]}^{m+1} \\ &\leq C_m \|g\|_{p-var, [0, T]}^p \sup_{|v-u| \leq |D|} \|g^m\|_{p-var, [u, v]}^{m+1-p} \rightarrow 0 \text{ as } |D| \rightarrow 0 \text{ (since } m+1 \geq [p]+1 > p). \end{aligned}$$

As a result,

$$\begin{aligned} g_t^{m+1} &= \int_0^t \beta_u(g_u^m) dg_u^m = \lim_{|D| \rightarrow 0, D = \{t_j\}_{j=0}^l \subset [0, t]} 1_m(g_{t_0, t_1}^m) \cdots 1_m(g_{t_{l-1}, t_l}^m) \\ &= \lim_{|D| \rightarrow 0, D \subset [0, t]} a_{m+1}^{t_0, t_1} a_{m+1}^{t_1, t_2} \cdots a_{m+1}^{t_{l-1}, t_l}, \quad \forall t \in [0, T]. \end{aligned}$$

Since $a_{m+1}^{s, t} \in \mathcal{G}_{m+1}$ and \mathcal{G}_{m+1} is closed, g^{m+1} takes values in \mathcal{G}_{m+1} . ■

3 Dominated paths

Let \mathcal{V} be a Banach space and suppose that $(T^{(n)}(\mathcal{V}), \mathcal{G}_n, \mathcal{P}_n)$ is a triple as in Notation 9. For Banach spaces E and F , let $L(E, F)$ denote the set of continuous linear mappings from E to F .

3.1 Structural assumptions on the group

Dominated paths are Banach-space valued paths that can be represented as integrals of time-varying cocyclic one-forms against a given group-valued path. We would like the set of dominated paths to be stable under some basic operations, which imposes some structural conditions on the group.

Condition 23 $T^{(n)}(\mathcal{V})$ is the smallest Banach space that includes \mathcal{G}_n , in the sense that, for Banach space \mathcal{U} and $\alpha \in L(T^{(n)}(\mathcal{V}), \mathcal{U})$, if $\alpha(g) = 0$, $\forall g \in \mathcal{G}_n$, then $\alpha(v) = 0$, $\forall v \in \mathcal{V}^{\otimes k}$, $k = 0, \dots, n$.

Condition 24 For $\sigma_i \in \mathcal{P}_n$, $i = 1, \dots, k$, satisfying $|\sigma_1| + |\sigma_2| + \cdots + |\sigma_k| \leq n$, there exists $\sigma_1 * \sigma_2 * \cdots * \sigma_k \in L(\mathcal{V}^{\otimes(|\sigma_1| + \cdots + |\sigma_k|)}, \mathcal{V}^{\otimes|\sigma_1|} \otimes \cdots \otimes \mathcal{V}^{\otimes|\sigma_k|})$ such that

$$(\sigma_1 * \sigma_2 * \cdots * \sigma_k)(a) = \sigma_1(a) \otimes \sigma_2(a) \otimes \cdots \otimes \sigma_k(a), \quad \forall a \in \mathcal{G}_n. \quad (3.1)$$

It is always possible to extend the algebra (and group) by adding in monomials of projective mappings so that Condition 24 holds. The product $*$ induces a coproduct on the Banach algebra $T^{(n)}(\mathcal{V})$, and the algebraic structure corresponds naturally to a Hopf algebra [15, 16].

We assume that $T^{(n)}(\mathcal{V})^{\otimes 2}$ is another Banach algebra, equipped with an admissible norm (see Definition 8) and with a multiplication, which is a continuous bilinear operator from $T^{(n)}(\mathcal{V})^{\otimes 2} \times T^{(n)}(\mathcal{V})^{\otimes 2}$ to $T^{(n)}(\mathcal{V})^{\otimes 2}$ satisfying

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (a_1 a_2) \otimes (b_1 b_2), \forall a_i, b_i \in T^{(n)}(\mathcal{V}).$$

Condition 25 *There exists a continuous linear mapping \mathcal{I} from $T^{(n)}(\mathcal{V})$ to $T^{(n)}(\mathcal{V})^{\otimes 2}$ satisfying*

$$\mathcal{I}(1) = \mathcal{I}(\mathcal{V}) = 0, \mathcal{I}(\mathcal{V}^{\otimes k}) \subseteq \sum_{j_1+j_2=k, j_i \geq 1} \mathcal{V}^{\otimes j_1} \otimes \mathcal{V}^{\otimes j_2}, k = 2, \dots, n. \quad (3.2)$$

In addition, let $1_{n,2}$ denote the projection of $T^{(n)}(\mathcal{V})^{\otimes 2}$ to $\sum_{j_1+j_2 \leq n, j_i \geq 1} \mathcal{V}^{\otimes j_1} \otimes \mathcal{V}^{\otimes j_2}$, then

$$\mathcal{I}(ab) = \mathcal{I}(a) + 1_{n,2}((a \otimes a)\mathcal{I}(b)) + 1_{n,2}((a-1) \otimes (a(b-1))), \forall a, b \in \mathcal{G}_n. \quad (3.3)$$

For a potential choice of the mapping \mathcal{I} , if for any $g \in C([0, T], \mathcal{G}_n)$, the “formal” integral

$$1_{n,2}(\iint_{0 < u_1 < u_2 < T} \delta g_{0,u_1} \otimes \delta g_{0,u_2})$$

is well-defined and can be represented as a universal continuous linear function of $g_{0,T}$, then define

$$\mathcal{I}(a) := 1_{n,2} \left(\iint_{0 < u_1 < u_2 < T} \delta(g_{0,u_1}) \otimes \delta(g_{0,u_2}) \right), \quad g \in C([0, T], \mathcal{G}_n), \quad g_{0,T} = a, \quad \forall a \in \mathcal{G}_n, \quad (3.4)$$

which extends linearly to $T^{(n)}(\mathcal{V})$. By “universal”, we mean that \mathcal{I} is independent of the selection of g and independent of a . In this formal definition, “ δ ” is comparable to the differential operator and “ \int ” is comparable to the integral operator. Normally, (3.3) follows from $\iint_{s < u_1 < u_2 < t} = \iint_{s < u_1 < u_2 < u} + \iint_{u < u_1 < u_2 < t} + \iint_{s < u_1 < u} \iint_{u < u_2 < t}$ for $s < u < t$, and (3.2) holds if $\delta(1) = 0$. Yet, both (3.2) and (3.3) have to be checked rigorously for a specific choice of the group.

The existence of the mapping \mathcal{I} imposes a stronger structural assumption on the group than that is needed for the rough integration. In rough integration, knowing how to integrate monomials against the degree-one monomial on paths space, we know how to integrate sufficiently smooth one-forms against the path [25, 14, 15]. The information needed for rough integration is encoded in the mapping \mathcal{I}' below.

Condition 25' *There exists a continuous linear mapping \mathcal{I}' from $T^{(n)}(\mathcal{V})$ to $T^{(n)}(\mathcal{V})^{\otimes 2}$ satisfying*

$$\mathcal{I}'(1) = \mathcal{I}'(\mathcal{V}) = 0, \mathcal{I}'(\mathcal{V}^{\otimes k}) \subseteq \mathcal{V}^{\otimes(k-1)} \otimes \mathcal{V}, k = 2, \dots, n.$$

In addition, let $1'_{n,2}$ denote the projection of $T^{(n)}(\mathcal{V})^{\otimes 2}$ to $\sum_{k=2}^n \mathcal{V}^{\otimes(k-1)} \otimes \mathcal{V}$, then

$$\mathcal{I}'(ab) = \mathcal{I}'(a) + 1'_{n,2}((a \otimes a)\mathcal{I}'(b)) + 1'_{n,2}((a-1) \otimes (a(b-1))), \quad \forall a, b \in \mathcal{G}_n.$$

The mapping \mathcal{I}' represents the formal integral $1_{n,2}(\iint_{0 < u_1 < u_2 < T} \delta g_{0,u_1} \otimes \delta x_{0,u_2}^1)$ with $x^1 := \pi_1(g)$ and contains part of the information of the mapping \mathcal{I} . The mapping \mathcal{I}' contains the information of how to integrate monomials against the degree-one monomial on paths space: $\iint_{0 < u_1 < u_2 < T} \delta x_{0,u_1}^k \otimes \delta x_{0,u_2}^1$, $x^k := \pi_k(g)$, and \mathcal{I}' is sufficient and necessary to define rough integration (Lemma 11 [27]). In the same manner, the mapping \mathcal{I} encodes the integration of a monomial against another monomial (not only the degree-one monomial): $\iint_{0 < u_1 < u_2 < T} \delta x_{0,u_1}^k \otimes \delta x_{0,u_2}^j$, and is sufficient and necessary to define the iterated integration for dominated paths (Proposition 32) and for controlled paths (Corollary 48).

The mapping \mathcal{I} resp. \mathcal{I}' are important because they identify algebraic properties of the group needed to define rough integration resp. iterated integration. They can be seen as counterparts of Chen’s identity (that encodes paths evolution) in paths integration. The mapping \mathcal{I} exists for step- n nilpotent Lie group $n \geq 1$ and step-2 Butcher group; the mapping \mathcal{I}' exists for step- n nilpotent Lie group and step- n Butcher group $n \geq 1$. In [27], the mapping \mathcal{I}' is employed to define Picard iterations for rough differential equations and prove the unique existence and continuity of the solution when the driving path lives in step- n nilpotent Lie group or step- n Butcher group for $n \geq 1$.

In Section 4, we prove that dominated paths are stable under (1) iterated integration, (2) multiplication, (3) composition with regular functions, and (4) is a transitive property. Condition 23 is used in all four properties; Condition 24 is used in (2) and (3); Condition 25 is used in (1) and (4).

3.1.1 Example: nilpotent Lie group

Conditions 23, 24 and 25 hold when \mathcal{G}_n is the step- n nilpotent Lie group over Banach space \mathcal{V} .

For Condition 23, suppose $\alpha \in L(T^{(n)}(\mathcal{V}), \mathcal{U})$ satisfies $\alpha(g) = 0, \forall g \in \mathcal{G}_n$. Then for $k = 1, \dots, n$ and $v_i \in \mathcal{V}, i = 1, \dots, k$, by considering the finite-dimensional space spanned by $\{v_i\}_{i=1}^k$ and by applying Poincaré-Birkhoff-Witt theorem, we have $\alpha(v_1 \otimes \dots \otimes v_k) = 0, \forall \{v_i\}_{i=1}^k \subset \mathcal{V}$, which implies $\alpha(v) = 0, \forall v \in \mathcal{V}^{\otimes k}$ (since $\mathcal{V}^{\otimes k}$ is the closure of the linear span of $\{v_1 \otimes \dots \otimes v_k | v_i \in \mathcal{V}, i = 1, \dots, k\}$ and α is a continuous linear mapping).

Condition 24 is satisfied by using the shuffle product (p36 [26]). Indeed, for $(k_1, k_2, \dots, k_l) \in \{1, \dots, n\}^l$,

$$\pi_{k_1}(a) \otimes \pi_{k_2}(a) \otimes \dots \otimes \pi_{k_l}(a) = \sum_{\varrho \in \text{Shuffles}(k_1, k_2, \dots, k_l)} \rho(\pi_{k_1+k_2+\dots+k_l}(a)),$$

where $\rho \in \text{Shuffles}(k_1, k_2, \dots, k_l)$ induces a continuous linear mapping from $\mathcal{V}^{\otimes(k_1+\dots+k_l)}$ to $\mathcal{V}^{\otimes k_1} \otimes \dots \otimes \mathcal{V}^{\otimes k_l}$.

For Condition 25, if we assume that any $g \in C([0, T], \mathcal{G}_n)$ satisfies the formal differential equations: $\delta(\pi_k(g_{0,t})) = \pi_{k-1}(g_{0,t}) \otimes \delta x_t, \forall t \in [0, T], k = 1, \dots, n$, with $x := \pi_1(g)$, then

$$\begin{aligned} & 1_{n,2} \left(\iint_{0 < u_1 < u_2 < T} \delta(g_{0,u_1}) \otimes \delta(g_{0,u_2}) \right) \\ &= 1_{n,2} \left(\int_0^T (g_{0,u} - 1) \otimes \delta(g_{0,u}) \right) = \sum_{k_1+k_2 \leq n, k_i \geq 1} \int_0^T \pi_{k_1}(g_{0,u}) \otimes \delta \pi_{k_2}(g_{0,u}) \\ &= \sum_{k_1+k_2 \leq n, k_i \geq 1} \int_0^T \pi_{k_1}(g_{0,u}) \otimes \pi_{k_2-1}(g_{0,u}) \otimes \delta x_u \\ &= \sum_{k_1+k_2 \leq n, k_i \geq 1} \sum_{\varrho \in \text{Shuffles}(k_1, k_2-1)} (\varrho, 1) (\pi_{k_1+k_2}(g_{0,T})). \end{aligned}$$

For $\varrho \in \text{Shuffles}(k_1, k_2-1)$, $(\varrho, 1)$ denotes the continuous linear mapping from $\mathcal{V}^{\otimes(k_1+k_2)}$ to $\mathcal{V}^{\otimes k_1} \otimes \mathcal{V}^{\otimes k_2}$ induced by $(\rho, 1)$ which is an element of the symmetric group of order (k_1+k_2) whose first (k_1+k_2-1) elements coincide with ρ with the last element unchanged. Then using (3.4), we obtain that

$$\mathcal{I}(v) := \sum_{k_1+k_2 \leq n, k_i \geq 1} \sum_{\varrho \in \text{Shuffles}(k_1, k_2-1)} (\varrho, 1) (\pi_{k_1+k_2}(v)), \forall v \in T^{(n)}(\mathcal{V}), \quad (3.5)$$

which is a universal continuous linear mapping. Using $(\varrho, 1) \in L(\mathcal{V}^{\otimes(k_1+k_2)}, \mathcal{V}^{\otimes k_1} \otimes \mathcal{V}^{\otimes k_2}), \forall k_i \geq 1, k_1+k_2 \leq n$, the mapping \mathcal{I} satisfies (3.2). Then we check that \mathcal{I} satisfies (3.3). Since \mathcal{G}_n is a closed topological group in $T^{(n)}(\mathcal{V})$ and $T^{(n)}(\mathcal{V})$ is the closure of the linear span of $\{v_1 \otimes \dots \otimes v_k | v_i \in \mathcal{V}, k = 1, \dots, n\}$, for any $a, b \in \mathcal{G}_n$, there exist $v_i \in \mathcal{V}, i \geq 1$, and $a_m, b_m \in \mathcal{G}_n(\text{span}(\{v_i\}_{i=1}^m))$, $m \geq 1$, such that $\lim_{m \rightarrow \infty} a_m = a$ and $\lim_{m \rightarrow \infty} b_m = b$. If we prove that (3.3) holds for a_m and b_m for any $m \geq 1$, then by using continuity we can prove that (3.3) holds for a and b . For this, fix $m \geq 1$, we treat $\{v_i\}_{i=1}^m$ as a basis of an m -dimensional space. Then by using Chow-Rashevskii connectivity Theorem, there exist two continuous bounded variation paths x_m and y_m on $[0, 1]$ taking values in $\text{span}(\{v_i\}_{i=1}^m)$ such that $S_n(x_m)_{0,1} = a_m$ and $S_n(y_m)_{0,1} = b_m$. Then $S_n(x_m)_{0,\cdot}$ and $S_n(y_m)_{0,\cdot}$ are two differentiable paths taking values in $\mathcal{G}_n(\text{span}(\{v_i\}_{i=1}^m))$ that satisfy

$$\begin{aligned} \mathcal{I}(a_m) &= 1_{n,2} \left(\iint_{0 < u_1 < u_2 < 1} dS_n(x_m)_{0,u_1} \otimes dS_n(x_m)_{0,u_2} \right), \\ \mathcal{I}(b_m) &= 1_{n,2} \left(\iint_{0 < u_1 < u_2 < 1} dS_n(y_m)_{0,u_1} \otimes dS_n(y_m)_{0,u_2} \right). \end{aligned}$$

Let $x_m \sqcup y_m : [0, 2] \rightarrow \text{span}(\{v_i\}_{i=1}^m)$ denote the concatenation of x_m and y_m . Then by using Chen's identity, we have $S_n(x_m \sqcup y_m)_{0,2} = a_m b_m$. By using the definition of the mapping \mathcal{I} , we have

$$\begin{aligned} \mathcal{I}(a_m b_m) &= 1_{n,2} \left(\iint_{0 < u_1 < u_2 < 2} dS_n(x_m \sqcup y_m)_{0,u_1} \otimes dS_n(x_m \sqcup y_m)_{0,u_2} \right) \\ &= 1_{n,2} \left(\iint_{0 < u_1 < u_2 < 1} + \iint_{1 < u_1 < u_2 < 2} + \int_{0 < u_1 < 1} \int_{1 < u_2 < 2} \right) \\ &= \mathcal{I}(a_m) + 1_{n,2}((a_m \otimes a_m) \mathcal{I}(b_m)) + 1_{n,2}((a_m - 1) \otimes (a_m(b_m - 1))), \end{aligned}$$

so (3.3) holds for a_m and b_m .

3.1.2 Example: Butcher group

Conditions 23 and 24 hold when \mathcal{G}_n is the step- n Butcher group over \mathbb{R}^d . Condition 25 holds when $n = 2$. For $n \geq 3$, it is hard to construct the mapping \mathcal{I} in Condition 25, but Condition 25' holds and

the mapping \mathcal{I}' exists. The mapping \mathcal{I}' encodes the integration of monomials against the degree-one monomial on paths space, and that is sufficient and necessary to define the rough integration [25, 14, 15].

Condition 23 holds for similar reasons as for the nilpotent Lie group.

For labelled forests $\sigma_i \in \mathcal{P}_n$, $i = 1, \dots, k$, satisfying $|\sigma_1| + \dots + |\sigma_k| \leq n$, $(\sigma_1 \dots \sigma_k)$ is again a labelled forest of degree less or equal to n , so $(\sigma_1 \dots \sigma_k) \in \mathcal{P}_n$ and $\sigma_1(a) \dots \sigma_k(a) = (\sigma^1 \dots \sigma^k)(a)$ for any $a \in \mathcal{G}_n$ based on (2.6). Hence, Condition 24 holds.

When $n = 2$, we can prove that Condition 25 holds. \mathcal{P}_2 is the set of labelled forests with degree less or equal to 2, i.e. $\mathcal{P}_2 = \{\bullet_i, \bullet_i \bullet_j, \mathbf{J}_i^j | i, j \in \{1, \dots, d\}\}$. The property of \mathcal{I} in (3.3) reduces to $(\{e_i\}_{i=1}^d)$ a basis of \mathbb{R}^d

$$\mathcal{I}(ab) = \mathcal{I}(a) + \mathcal{I}(b) + \sum_{i,j=1}^d (\bullet_j(a)) e_j \otimes (\bullet_i(b)) e_i, \forall a, b \in \mathcal{G}_2. \quad (3.6)$$

Then, let $\mathcal{I}(a) := \sum_{i,j=1}^d \mathbf{J}_i^j(a) e_j \otimes e_i$, that is a universal continuous linear mapping from $T^{(2)}(\mathbb{R}^d)$ to $\mathbb{R}^d \otimes \mathbb{R}^d$. Then the property (3.6) holds because, based on the rule of multiplication in the Butcher group, $\mathbf{J}_i^j(ab) = \mathbf{J}_i^j(a) + \mathbf{J}_i^j(b) + (\bullet_j(a))(\bullet_i(b))$, $\forall a, b \in \mathcal{G}_2$. Equivalently, we could assume that any $g \in C([0, T], \mathcal{G}_2)$ satisfies the formal differential equation $\delta(\mathbf{J}_i^j(g_{0,t})) = (\bullet_j(g_{0,t}))\delta x_t^i$ with $x_t^i := \bullet_i(g_{0,t})$. Then, for $a \in \mathcal{G}_2$ and $g \in C([0, T], \mathcal{G}_2)$ satisfying $g_{0,T} = a$, we have

$$\begin{aligned} \mathcal{I}(a) &= 1_{n,2} \left(\iint_{0 < u_1 < u_2 < T} \delta(g_{0,u_1}) \otimes \delta(g_{0,u_2}) \right) \\ &= \sum_{i,j=1}^d \int_0^T (\bullet_j(g_{0,u})) \delta x_u^i e_j \otimes e_i = \sum_{i,j=1}^d \mathbf{J}_i^j(g_{0,T}) e_j \otimes e_i = \sum_{i,j=1}^d \mathbf{J}_i^j(a) e_j \otimes e_i. \end{aligned} \quad (3.7)$$

The existence of \mathcal{I} when $n = 2$ gives an explanation to the existence of the canonical enhancement of a controlled path when $2 \leq p < 3$ (see Theorem 1 [14] and Corollary 48 below).

For $n \geq 3$, the problem is complicated and finding a mapping \mathcal{I} satisfying Condition 25 is difficult. Indeed, for $g \in C([0, T], \mathcal{G}_3)$, it is hard to represent

$$\int_0^T (\bullet_k(g_{0,t})) \delta((\bullet_i \bullet_j)(g_{0,t})), (i, j, k) \in \{1, \dots, d\}^3,$$

as a linear functional of $g_{0,T}$, because the integration by parts formula does not hold. For $n \geq 3$, Condition 25' holds. If we assume (as in Theorem 8.5 [15]) that any $g \in C([0, T], \mathcal{G}_n)$ satisfies the formal differential equations that, for any labelled forest $\sigma \in \mathcal{P}_{n-1}$,

$$\delta[\sigma]_i(g_{0,t}) = \sigma(g_{0,t}) \delta x_t^i, \quad \forall t \in [0, T], \text{ with } x_t^i := \bullet_i(g_{0,t}),$$

where $[\sigma]_i$ denotes the labelled tree obtained by attaching σ to a new root with label i . Then, we can define $\mathcal{I}' \in L(T^{(n)}(\mathbb{R}^d), T^{(n)}(\mathbb{R}^d)^{\otimes 2})$ by, for any $a \in \mathcal{G}_n$ and $g \in C([0, T], \mathcal{G}_n)$ satisfying $g_{0,T} = a$,

$$\begin{aligned} \mathcal{I}'(a) &: = 1_{n,2} \left(\int_0^T (g_{0,u} - 1) \otimes \delta x_u \right) \\ &= \sum_{i=1}^d \sum_{\sigma \in \mathcal{P}_{n-1}, |\sigma|=1, \dots, n-1} \int_0^T \sigma(g_{0,u}) \delta x_u^i e_\sigma \otimes e_i \\ &= \sum_{i=1}^d \sum_{\sigma \in \mathcal{P}_{n-1}, |\sigma|=1, \dots, n-1} [\sigma]_i(a) e_\sigma \otimes e_i, \end{aligned} \quad (3.8)$$

where $e_\sigma \in (\mathbb{R}^d)^{\otimes |\sigma|}$ is the basis coordinate corresponding to σ and $e_\sigma \otimes e_i$ is treated as an element in $(\mathbb{R}^d)^{\otimes |\sigma|} \otimes \mathbb{R}^d \subset T^{(n)}(\mathbb{R}^d)^{\otimes 2}$. Hence, \mathcal{I}' is a universal continuous linear mapping from $T^{(n)}(\mathbb{R}^d)$ to $T^{(n)}(\mathbb{R}^d)^{\otimes 2}$ that satisfies $\mathcal{I}'(1) = \mathcal{I}'(\mathbb{R}^d) = 0$, $\mathcal{I}'((\mathbb{R}^d)^{\otimes k}) \subseteq (\mathbb{R}^d)^{\otimes(k-1)} \otimes \mathbb{R}^d$, $k = 2, \dots, n$. Moreover, by using the multiplication in Butcher group, we have, for $a, b \in \mathcal{G}_n$ and $\sigma \in \mathcal{P}_{n-1}$ satisfying $\Delta\sigma = \sum_j \sigma^{1,j} \otimes \sigma^{2,j}$,

$$[\sigma]_i(ab) = [\sigma]_i(a) + \sum_j \sigma^{1,j}(a) [\sigma^{2,j}]_i(b), \quad \forall a, b \in \mathcal{G}_n.$$

This implies that,

$$\begin{aligned} &\sum_{\sigma \in \mathcal{P}_{n-1}, |\sigma|=0, \dots, n-1} [\sigma]_i(ab) e_\sigma \otimes e_i \\ &= \sum_{\sigma \in \mathcal{P}_{n-1}, |\sigma|=0, \dots, n-1} [\sigma]_i(a) e_\sigma \otimes e_i + (a \otimes 1) \left(\sum_{\sigma \in \mathcal{P}_{n-1}, |\sigma|=0, \dots, n-1} [\sigma]_i(b) e_\sigma \otimes e_i \right), \end{aligned} \quad (3.9)$$

where $e_\sigma \otimes e_i$ is treated as an element in $(\mathbb{R}^d)^{\otimes |\sigma|} \otimes \mathbb{R}^d \subset T^{(n)}(\mathbb{R}^d)^{\otimes 2}$. Hence, if we let $1'_{n,2}$ denote the projection of $T^{(n)}(\mathbb{R}^d)^{\otimes 2}$ to $\sum_{k=2}^n (\mathbb{R}^d)^{\otimes(k-1)} \otimes \mathbb{R}^d$, then based on (3.8) and (3.9) we have

$$\begin{aligned} \mathcal{I}'(ab) &= \mathcal{I}'(a) + 1'_{n,2}((a \otimes 1) \mathcal{I}'(b)) + 1'_{n,2}((a - 1) \otimes (\bullet(b))) \\ &= \mathcal{I}'(a) + 1'_{n,2}((a \otimes a) \mathcal{I}'(b)) + 1'_{n,2}((a - 1) \otimes (a(b - 1))), \quad \forall a, b \in \mathcal{G}_n, \end{aligned}$$

where the term $1'_{n,2}((a-1) \otimes (a(b-1)))$ is caused by the different ranges of summation of $\sigma \in \mathcal{P}_n$ in (3.8) and (3.9).

It is hard to find a mapping \mathcal{I} for step- n Butcher group $n \geq 3$, because it is hard to differentiate a path taking values in the group. Unlike the nilpotent Lie group where any group-valued path g satisfies the ‘formal’ differential equation $\delta g = g \delta x$ with $x := \pi_1(g)$, in this case it is hard to find a continuous mapping F such that any path taking values in step- n Butcher group $n \geq 3$ satisfies $\delta g = F(g) \delta x$. The mapping \mathcal{I} exists for step-2 Butcher group because we only need to define $\int x \delta x$ and to integrate against the degree-one monomial. For step- n Butcher group $n \geq 3$, to define the mapping \mathcal{I} , we need to define the differentiation of products that may easily drift out of the group as in the case of Itô differential equations.

3.2 Definition of dominated paths

Notation 26 Let \mathcal{U} be a Banach space and $\alpha \in L(T^{(n)}(\mathcal{V}), \mathcal{U})$. We denote

$$\begin{aligned} \|\alpha(\cdot)\| &:= \sup_{v \in T^{(n)}(\mathcal{V}), \|v\|=1} \|\alpha(v)\|, \\ \|\alpha(\cdot)\|_k &:= \sup_{v \in \mathcal{V}^{\otimes k}, \|v\|=1} \|\alpha(v)\|, \quad k = 0, 1, \dots, n. \end{aligned}$$

With $[p]$ we denote the largest integer that is less or equal to $p \geq 1$. We work with the triple $(T^{([p])}(\mathcal{V}), \mathcal{G}_{[p]}, \mathcal{P}_{[p]})$ and continuous paths of finite p -variation taking values in $\mathcal{G}_{[p]}$ i.e. $C^{p-var}([0, T], \mathcal{G}_{[p]})$.

Condition 27 (Slowly-Varying Condition) Suppose that $(T^{([p])}(\mathcal{V}), \mathcal{G}_{[p]}, \mathcal{P}_{[p]})$ satisfies Conditions 23, 24 and 25, $g \in C^{p-var}([0, T], \mathcal{G}_{[p]})$ and \mathcal{U} is a Banach space. We say $\beta \in C([0, T], B(\mathcal{G}_{[p]}, \mathcal{U}))$ is slowly-varying, if there exist $M > 0$, control ω and $\theta > 1$ such that

$$\begin{aligned} \|\beta_t(g_t, \cdot)\| &\leq M, \quad \forall t \in [0, T], \\ \|(\beta_t - \beta_s)(g_t, \cdot)\|_k &\leq \omega(s, t)^{\theta - \frac{k}{p}}, \quad \forall 0 \leq s < t \leq T, \quad k = 1, 2, \dots, [p]. \end{aligned}$$

We define the operator norm of β by

$$\|\beta\|_\theta^\omega := \sup_{t \in [0, T]} \|\beta_t(g_t, \cdot)\| + \max_{k=1, \dots, [p]} \sup_{0 \leq s < t \leq T} \frac{\|(\beta_t - \beta_s)(g_t, \cdot)\|}{\omega(s, t)^{\theta - \frac{k}{p}}}. \quad (3.10)$$

The norm $\|\cdot\|_\theta^\omega$ is used in [27] to quantify the convergence of one-forms associated with Picard iterations for rough differential equations.

If β satisfies the slowly-varying condition for g , then (β, g) satisfies the integrable condition (Condition 14). Indeed, for $s < u < t$,

$$\begin{aligned} \|(\beta_u - \beta_s)(g_u, g_{u,t})\| &\leq \sum_{\sigma \in \mathcal{P}_{[p]}} \|(\beta_u - \beta_s)(g_u, \sigma(g_{u,t}))\| \\ &\leq \sum_{\sigma \in \mathcal{P}_{[p]}} \|(\beta_u - \beta_s)(g_u, \cdot)\|_{|\sigma|} \|\sigma(g_{u,t})\| \\ &\leq \sum_{\sigma \in \mathcal{P}_{[p]}} \omega(s, u)^{\theta - \frac{|\sigma|}{p}} \|g\|_{p-var, [u, t]}^{|\sigma|} \leq C_p(\omega(s, t) + \|g\|_{p-var, [s, t]}^p)^\theta. \end{aligned}$$

Definition 28 (Dominated Paths) Suppose $g \in C^{p-var}([0, T], \mathcal{G}_{[p]})$, and for a Banach space \mathcal{U} , $h \in C([0, T], \mathcal{U})$. If there exists $\beta \in C([0, T], B(\mathcal{G}_{[p]}, \mathcal{U}))$ that is slowly-varying and satisfies

$$h_t = h_0 + \int_0^t \beta_u(g_u) dg_u, \quad \forall t \in [0, T]. \quad (3.11)$$

Then we call h a path dominated by g with the one-form β , and refer to the process of starting with the path h and fixing a choice of the one-form β as coupling h to g via β to make a dominated path.

Using Theorem 15, for control $\hat{\omega} := \omega + \|g\|_{p-var}^p$ and $\theta > 1$, we have

$$\|h_t - h_s - \beta_s(g_s, g_{s,t})\| \leq C_{p, \theta, \hat{\omega}(0, T)} \|\beta\|_\theta^\omega \hat{\omega}(s, t)^\theta, \quad \forall s < t, \quad (3.12)$$

and the function $(\beta, \|\cdot\|_\theta^\omega) \mapsto (h, \|\cdot\|_{p-var, [0, T]})$ is a Lipschitz function:

$$\|h\|_{p-var, [0, T]} \leq C_{p, \theta, \hat{\omega}(0, T)} \|\beta\|_\theta^\omega. \quad (3.13)$$

3.3 Weakly controlled paths and dominated paths

Dominated paths bear some similarities to controlled paths [14, 15]. We quote the definition of κ -weakly controlled paths in Def 8.1 [15]:

Definition 29 (Weakly Controlled Paths, Gubinelli) *Let X be a γ -BRP and let n be the largest integer such that $n\gamma \leq 1$. For $\kappa \in (1/(n+1), \gamma]$ a path y is a κ -weakly controlled by X with values in V if there exist paths $\{y^\tau \in \mathcal{C}_1^{|\tau|\kappa}(V) : \tau \in \mathcal{F}_{\mathcal{L}}^{n-1}\}$ and remainders $\{y^\# \in \mathcal{C}_2^{n\kappa}(V), y^{\#,\tau} \in \mathcal{C}_2^{(n-|\tau|)\kappa}(V), \tau \in \mathcal{F}_{\mathcal{L}}^{n-1}\}$ such that*

$$\delta y = \sum_{\tau \in \mathcal{F}_{\mathcal{L}}^{n-1}} X^\tau y^\tau + y^\# \quad (3.14)$$

and for $\tau \in \mathcal{F}_{\mathcal{L}}^{n-1}$:

$$\delta y^\tau = \sum_{\sigma \in \mathcal{F}_{\mathcal{L}}^{n-1}} \sum_{\rho} c'(\sigma, \tau, \rho) X^\rho y^\sigma + y^{\tau,\#} \quad (3.15)$$

where we mean $\delta y^\tau = y^{\tau,\#}$ when $|\tau| = n-1$.

Note that in a convenient abuse of language, although it is the coupling y^τ of y to X that is the weakly controlled path, it is customary to write as if the symbol y alone was the weakly controlled path! It is a good cautionary exercise to give examples of a non zero coupling of the zero path to X .

Translated to our language, Definition 29 can be rewritten as follows. For $p \geq 1$, suppose $\mathcal{G}_{[p]}$ is the step- $[p]$ Butcher group over \mathbb{R}^d , $\mathcal{P}_{[p]-1}$ denotes the set of labelled forests of degree less or equal to $[p]-1$ and $g \in C^{p-var}([0, T], \mathcal{G}_{[p]})$. Then for Banach space \mathcal{U} , $\gamma \in C([0, T], \mathcal{U})$ is a path controlled by g , if there exist a family of paths $\gamma^\sigma \in C([0, T], \mathcal{U})$ indexed by $\sigma \in \mathcal{P}_{[p]-1}$, $|\sigma| \geq 1$, and constants $\theta > 1$, $C > 0$, such that, γ satisfies

$$\left\| \gamma_t - \gamma_s - \sum_{\sigma \in \mathcal{P}_{[p]-1}, |\sigma| \geq 1} \gamma_s^\sigma \sigma(g_{s,t}) \right\| \leq C(\|g\|_{p-var, [s,t]}^p)^{\theta - \frac{1}{p}}, \quad \forall 0 \leq s < t \leq T, \quad (3.16)$$

and γ^σ , $\sigma \in \mathcal{P}_{[p]-1}$, $|\sigma| \geq 1$, satisfies

$$\left\| \gamma_t^\sigma - \gamma_s^\sigma - \sum_{\sigma_i \in \mathcal{P}_{[p]-1}, |\sigma_i| \geq 1} c'(\sigma_1, \sigma_2, \sigma) \gamma_s^{\sigma_1} \sigma_2(g_{s,t}) \right\| \leq C(\|g\|_{p-var, [s,t]}^p)^{\theta - \frac{1+|\sigma|}{p}}, \quad \forall 0 \leq s < t \leq T, \quad (3.17)$$

where $c'(\sigma_1, \sigma_2, \sigma)$ counts the number of $\sigma_2 \otimes \sigma$ in the reduced comultiplication $\Delta' \sigma_1 = \Delta \sigma_1 - \sigma_0 \otimes \sigma_1 - \sigma_1 \otimes \sigma_0$ of σ_1 (with σ_0 denoting the projection to \mathbb{R}).

We can redefine controlled paths by using time-varying cocyclic one-forms.

Definition 30 (Weakly Controlled Paths) *Let $g \in C^{p-var}([0, T], \mathcal{G}_{[p]})$ and let \mathcal{U} be a Banach space. We say that $\gamma : [0, T] \rightarrow \mathcal{U}$ is a path weakly controlled by g , if there exist control ω and $\beta : [0, T] \rightarrow B(\mathcal{G}_{[p]-1}, \mathcal{U})$ satisfying*

$$\|\gamma_t - \gamma_s - \beta_s(g_s, g_{s,t})\| \leq \omega(s, t)^{\theta - \frac{1}{p}}, \quad \forall 0 \leq s < t \leq T, \quad (3.18)$$

$$\|(\beta_t - \beta_s)(g_t, \cdot)\|_k \leq \omega(s, t)^{\theta - \frac{1+k}{p}}, \quad \forall 0 \leq s < t \leq T, \quad k = 1, \dots, [p]-1. \quad (3.19)$$

If γ is a controlled path in the sense of Definition 29 then γ is a controlled path in the sense of Definition 30. Indeed, we can rewrite (3.16) and (3.17) in term of time-varying cocyclic one-forms. Define $\beta \in C([0, T], B(\mathcal{G}_{[p]-1}, \mathcal{U}))$ by

$$\beta_s(a, b) := \sum_{\sigma \in \mathcal{P}_{[p]-1}, |\sigma| \geq 1} \gamma_s^\sigma \sigma(g_s^{-1} a (b - \sigma_0(b))), \quad \forall a, b \in \mathcal{G}_{[p]-1}, \quad \forall 0 \leq s \leq T.$$

Then (3.16) \Leftrightarrow (3.18), and (3.17) \Rightarrow (3.19). Indeed, (3.17) implies (3.19), because for any $a \in \mathcal{G}_{[p]-1}$ we have

$$\begin{aligned} (\beta_t - \beta_s)(g_t, a) &= \sum_{\sigma \in \mathcal{P}_{[p]-1}, |\sigma| \geq 1} \gamma_t^\sigma \sigma(a - \sigma_0(a)) - \sum_{\sigma \in \mathcal{P}_{[p]-1}, |\sigma| \geq 1} \gamma_s^\sigma \sigma(g_{s,t}(a - \sigma_0(a))) \\ &= \sum_{\sigma \in \mathcal{P}_{[p]-1}, |\sigma| \geq 1} \left(\gamma_t^\sigma - \gamma_s^\sigma - \sum_{\sigma_i \in \mathcal{P}_{[p]-1}, |\sigma_i| \geq 1} c'(\sigma_1, \sigma_2, \sigma) \gamma_s^{\sigma_1} \sigma_2(g_{s,t}) \right) \sigma(a), \end{aligned} \quad (3.20)$$

where the constant $c'(\sigma_1, \sigma_2, \sigma)$ is defined as in (3.17). Since both sides of (3.20) are linear in a , based on Condition 23 (we proved that it holds for Butcher group), (3.20) holds for any $v \in (\mathbb{R}^d)^{\otimes k}$, $k = 1, \dots, [p]-1$. Hence,

$$\begin{aligned} \|(\beta_t - \beta_s)(g_t, \cdot)\|_k &\leq \sum_{\sigma \in \mathcal{P}_{[p]-1}, |\sigma| = k} \left\| \left(\gamma_t^\sigma - \gamma_s^\sigma - \sum_{\sigma_i \in \mathcal{P}_{[p]-1}, |\sigma_i| \geq 1} c'(\sigma_1, \sigma_2, \sigma) \gamma_s^{\sigma_1} \sigma_2(g_{s,t}) \right) \sigma(\cdot) \right\|_k \\ &\leq C(\|g\|_{p-var, [s,t]}^p)^{\theta - \frac{1+k}{p}}, \quad \forall 0 \leq s < t \leq T, \quad k = 1, \dots, [p]-1. \end{aligned}$$

The space of controlled paths is a linear space and is preserved under composition with regular functions. Moreover, when $2 \leq p < 3$, for paths γ^1 and γ^2 controlled by $g \in C^{p-var}([0, T], \mathcal{G}_{[p]})$, the integral path $\int_0^\cdot \gamma_u^1 \otimes d\gamma_u^2$ is canonically defined and is again a path controlled by g (Theorem 1 [14]). When $p \geq 3$, for path γ controlled by $g \in C^{p-var}([0, T], \mathcal{G}_{[p]})$ with x denoting the first level of g , the integral path $\int_0^\cdot \gamma_u \otimes dx_u$ is well defined and is again a path controlled by g (Theorem 8.5 [15], Definition 4.17 [10]). The existence of the canonical integral of controlled paths when $2 \leq p < 3$ resp. $p \geq 3$ is closely related to the mappings \mathcal{I} in (3.7) resp. \mathcal{I}' in (3.8) (see also Remark 34 and Corollary 48 below).

Comparing with the definition of dominated paths, we have

Proposition 31 *If γ is a path dominated by g , then γ is a path weakly controlled by g as in Definition 30.*

The other direction is not necessarily true, and the one-form associated with a controlled path can vary a little quicker than the one-form associated with a dominated path. Roughly speaking, the relationship between the controlled path and the dominated path is comparable to the relationship between the integrand and the indefinite integral got after integration or to the difference between a weak and a strong solution to a stochastic differential equation. The slowly varying one form in the coupling realises γ as a function of X .

In [27] we proved that, when the mapping \mathcal{I}' exists (e.g. step- n nilpotent Lie group or step- n Butcher group $n \geq 1$), for controlled path γ and $x := \pi_1(g)$, the indefinite integral $\int_0^\cdot \gamma_u \otimes dx_u$ is well-defined as a dominated path. More generally, when the mapping \mathcal{I} exists (e.g. step- n nilpotent Lie group $n \geq 1$ or step-2 Butcher group), for controlled path γ^1 and dominated path γ^2 , the integral path $\int_0^\cdot \gamma_u^1 \otimes d\gamma_u^2$ is well defined and is another dominated path (Remark 34).

In the definition of dominated paths e.g. $h = h_0 + \int_0^\cdot \beta_u(g_u) dg_u$ in (3.11), β is integrable against g and h is determined by β . Indeed, dominated paths are all about integrable one-forms, and the path is determined by the one-form. On the other hand, based on (3.18) and (3.19), for a controlled path γ , β does not necessarily satisfy the integrable condition, and γ is not uniquely determined by β . Indeed, for β satisfying (3.19), there does not necessarily exist a path γ satisfies (3.18); if there exists a γ satisfies (3.18), then $\gamma + \eta$ also satisfies (3.18) for any $\eta : [0, T] \rightarrow \mathcal{U}$ satisfying $\|\eta_t - \eta_s\| \leq C\|g\|_{p-var, [s, t]}^{p-1}$, $\forall 0 \leq s < t \leq T$. That the time-varying one-form is not sufficiently integrable and that the path is not uniquely determined by the one-form will always be there for a controlled path, which make the existence of the canonical iterated integral (when $2 \leq p < 3$ [14]) of two controlled paths a very interesting result. In fact the existence of the iterated integral is not solely about the one-form; it is the result of the interplay between the one-form and the path via the intermediary of integration (Section 2.2). Indeed, based on Corollary 48, for paths γ^i , $i = 1, 2$, controlled by $g \in C^{p-var}([0, T], \mathcal{G}_{[p]})$, $2 \leq p < 3$, the path $\int_0^\cdot \gamma_u^1 \otimes d\gamma_u^2$ can be represented as the integral of a time-varying cocyclic one-form against the group-valued path $\gamma^2 \oplus g$. As a result, for a path $\gamma : [0, T] \rightarrow \mathbb{R}^e$ controlled by $g \in C^{p-var}([0, T], \mathcal{G}_{[p]})$, $2 \leq p < 3$, there exists a canonical enhancement of γ to a path taking values in the step-2 Butcher group over \mathbb{R}^e :

$$\Gamma_t := 1 + \sum_{\sigma \in \mathcal{P}_2} x(\sigma)_t e_\sigma,$$

$$\text{with } x(\cdot)_t := \gamma_t^i - \gamma_0^i, \quad x(\cdot)_i \cdot x(\cdot)_j := x(\cdot)_i x(\cdot)_j, \quad x\left(\begin{smallmatrix} j \\ i \end{smallmatrix}\right)_t := \int_0^t x(\cdot)_j \otimes d\gamma_u^i, \quad \forall (i, j) \in \{1, \dots, e\}^2, \forall t,$$

where $e_\sigma \in (\mathbb{R}^e)^{\otimes |\sigma|}$ is the basis coordinate corresponding to the labelled forest σ . The set of paths dominated by Γ clearly includes γ . When γ is dominated by g , the set of paths dominated by Γ is a subset of the paths dominated by g (Proposition 40). Intuitively, one could split the space of controlled paths to subspaces of dominated paths (dominated by a slightly perturbed group-valued path). Each subspace is a linear space and an algebra, stable under iterated integration and composition with regular functions. It is also possible to union finitely many of these subspaces, which will be dominated by the joint signature of these controlled paths. The canonical enhancement of a controlled path is well-defined when the group is the step- n nilpotent Lie group for $n \geq 1$ or the step-2 Butcher group. While it should be noted that for step- n Butcher group, $n \geq 3$, it is difficult to construct the mapping \mathcal{I} in Condition 25, so it is difficult to define the enhancement of a controlled path in that case.

Working with dominated paths has the benefit that basic operations are *continuous* in operator norm in the space of one-forms (Section 4). For example, the one-form associated with the group-valued enhancement of a dominated path (built by using operations in Section 4, can take values in e.g. nilpotent Lie group or Butcher group) is continuous in operator norm w.r.t. the one-form associated with the base dominated path.

4 Stability of dominated paths

The set of dominated paths is a linear space, stable under iterated integration, is an algebra and is stable under composition with regular functions. Moreover, being a dominated path is a transitive property: if γ is dominated by g and we enhance γ via iterated integration to a group-valued path Γ , then those paths dominated by Γ form a subset of the paths dominated by g . All of these operations are continuous in associated one-forms, and explicit dependence in operator norm is given.

Coefficients in this section may depend on the norm of the mappings $\sigma_1 * \dots * \sigma_k$ (as in Condition 24) and the norm of the mapping \mathcal{I} (as in Condition 25).

Let \mathcal{U} be a Banach space and $\beta \in B(\mathcal{G}_{[p]}, \mathcal{U})$. Using that $\beta(a, b(c-1)) = \beta(ab, c) = \beta(ab, c-1)$, $\forall a, b, c \in \mathcal{G}_{[p]}$, the linearity in $c-1 = c - \sigma_0(c)$ and that $\mathbb{R} \oplus \mathcal{V} \oplus \dots \oplus \mathcal{V}^{\otimes [p]}$ is the linear span of $\mathcal{G}_{[p]}$ (Condition 23), we get

$$\beta(a, bv) = \beta(ab, v), \quad \forall a, b \in \mathcal{G}_{[p]}, \forall v \in \mathcal{V}^{\otimes k}, \quad k = 1, \dots, [p]. \quad (4.1)$$

4.1 Iterated integration

Recall the mapping $\mathcal{I} \in L(T^{([p])}(\mathcal{V}), T^{([p])}(\mathcal{V})^{\otimes 2})$ in Condition 25.

Proposition 32 (Iterated Integration) *Let $g \in C^{p-var}([0, T], \mathcal{G}_{[p]})$ and \mathcal{U}^i , $i = 1, 2$, be Banach spaces. Suppose $\int_0^\cdot \beta_u^i(g_u) dg_u : [0, T] \rightarrow \mathcal{U}^i$, $i = 1, 2$, are two dominated paths satisfying $\|\beta^i\|_{\theta_i}^{\omega_i} < \infty$ for control ω_i and $\theta_i > 1$, $i = 1, 2$ (as defined in (3.10)). Then there exists a dominated path $\int_0^\cdot \beta_u(g_u) dg_u : [0, T] \rightarrow \mathcal{U}^1 \otimes \mathcal{U}^2$ such that with control $\omega := \omega_1 + \omega_2 + \|g\|_{p-var}^p$ and $\theta := \min(\theta_1, \theta_2)$,*

$$\|\beta\|_\theta^\omega \leq C_{p, \omega(0, T)} \|\beta^1\|_{\theta_1}^{\omega_1} \|\beta^2\|_{\theta_2}^{\omega_2}, \quad (4.2)$$

and for $0 \leq s \leq t \leq T$,

$$\left\| \int_s^t \beta_u(g_u) dg_u - \int_s^t \beta_u^1(g_u) dg_u \otimes \beta_s^2(g_s, g_{s,t}) - \beta_s^1(g_s, \cdot) \otimes \beta_s^2(g_s, \cdot) \mathcal{I}(g_{s,t}) \right\| \leq \omega(s, t)^\theta, \quad (4.3)$$

where $\beta_s^1(g_s, \cdot) \otimes \beta_s^2(g_s, \cdot)$ denotes the unique continuous linear operator from $T^{([p])}(\mathcal{V})^{\otimes 2}$ to $\mathcal{U}^1 \otimes \mathcal{U}^2$ satisfying $(\beta_s^1(g_s, \cdot) \otimes \beta_s^2(g_s, \cdot))(v_1 \otimes v_2) = \beta_s^1(g_s, v_1) \otimes \beta_s^2(g_s, v_2)$, $\forall v_1, v_2 \in T^{([p])}(\mathcal{V})$.

Remark 33 Let $\gamma^i := \int_0^\cdot \beta_u^i(g_u) dg_u$, $i = 1, 2$. Then for $s < t$,

$$\begin{aligned} & \iint_{0 < u_1 < u_2 < t} d\gamma_{u_1}^1 \otimes d\gamma_{u_2}^2 - \iint_{0 < u_1 < u_2 < s} d\gamma_{u_1}^1 \otimes d\gamma_{u_2}^2 \\ &= (\gamma_s^1 - \gamma_0^1) \otimes (\gamma_t^2 - \gamma_s^2) + \iint_{s < u_1 < u_2 < t} d\gamma_{u_1}^1 \otimes d\gamma_{u_2}^2, \end{aligned}$$

with $(\gamma_s^1 - \gamma_0^1) \otimes (\gamma_t^2 - \gamma_s^2)$ and $\iint_{s < u_1 < u_2 < t} d\gamma_{u_1}^1 \otimes d\gamma_{u_2}^2$ correspond to the two parts in (4.3).

Remark 34 When γ^1 is a controlled path (Definition 30) and γ^2 is a dominated path, same proof applies and obtains that $\int_0^\cdot \gamma^1 \otimes d\gamma^2$ is a dominated path. In particular, when γ^1 is a controlled path and $\gamma^2 = \pi_1(g) := x$ (dominated by g with $\beta_s(a, b) = \pi_1(a(b-1))$ for all s), the indefinite integral $\int_0^\cdot \gamma^1 \otimes dx$ is a dominated path. This integral $\int_0^\cdot \gamma^1 \otimes dx$ is well-defined when Condition 25' (instead of Condition 25) is satisfied (see Lemma 11 [27]).

Proof. We check that β satisfies the slowly-varying condition (Condition 27). Then (4.3) follows from Theorem 15. We define $\beta^{1,2} \in C([0, T], C(\mathcal{G}_{[p]}, L(T^{([p])}(\mathcal{V}), \mathcal{U}^1 \otimes \mathcal{U}^2)))$ by, for $a \in \mathcal{G}_{[p]}$ and $v \in T^{([p])}(\mathcal{V})$,

$$\beta_s^{1,2}(a, v) := (\beta_s^1(a, \cdot) \otimes \beta_s^2(a, \cdot)) \mathcal{I}(v), \quad (4.4)$$

and define $\beta \in C([0, T], B(\mathcal{G}_{[p]}, \mathcal{U}^1 \otimes \mathcal{U}^2))$ by, for $a, b \in \mathcal{G}_{[p]}$ and $s \in [0, T]$,

$$\beta_s(a, b) := \int_0^s \beta_u^1(g_u) dg_u \otimes \beta_s^2(g_s, g_s^{-1}a(b-1)) + \beta_s^{1,2}(g_s, g_s^{-1}a(b-1)).$$

Recall the mapping \mathcal{I} in Condition 25 satisfy

$$\mathcal{I}(1) = \mathcal{I}(\mathcal{V}) = 0 \quad \text{and} \quad \mathcal{I}(\mathcal{V}^{\otimes k}) \subseteq \sum_{j_1+j_2=k, j_i \geq 1} \mathcal{V}^{\otimes j_1} \otimes \mathcal{V}^{\otimes j_2}, \quad k = 2, \dots, [p], \quad (4.5)$$

$$\mathcal{I}(ab) = \mathcal{I}(a) + 1_{[p],2}((a \otimes a) \mathcal{I}(b)) + 1_{[p],2}((a-1) \otimes (a(b-1))), \quad \forall a, b \in \mathcal{G}_{[p]}, \quad (4.6)$$

where $1_{[p],2}$ denotes the projection of $T^{([p])}(\mathcal{V})^{\otimes 2}$ to $\sum_{j_1+j_2 \leq [p], j_i \geq 1} \mathcal{V}^{\otimes j_1} \otimes \mathcal{V}^{\otimes j_2}$.

Fix $s < t$. Using (4.5), $\|(\beta_t^{1,2} - \beta_s^{1,2})(g_t, \cdot)\|_1 = 0$. Using (4.6), for $k = 2, \dots, [p]$ and $\theta = \min(\theta_1, \theta_2)$,

$$\begin{aligned} & \left\| \left(\beta_t^{1,2} - \beta_s^{1,2} \right) (g_t, \cdot) \right\|_k \\ & \leq C \sum_{j=1}^{k-1} \left(\left\| (\beta_t^1 - \beta_s^1)(g_t, \cdot) \right\|_j \left\| \beta_t^2(g_t, \cdot) \right\|_{k-j} + \left\| \beta_s^1(g_t, \cdot) \right\|_j \left\| (\beta_t^2 - \beta_s^2)(g_t, \cdot) \right\|_{k-j} \right) \\ & \leq C_{p,\omega(0,T)} \left\| \beta^1 \right\|_{\theta_1}^{\omega_1} \left\| \beta^2 \right\|_{\theta_2}^{\omega_2} \omega(s, t)^{\theta - \frac{k-1}{p}}. \end{aligned} \quad (4.7)$$

Moreover, by (4.6), for $a, b, c \in \mathcal{G}_{[p]}$ and $s \in [0, T]$, we have

$$\begin{aligned} \beta_s^{1,2}(a, bc) &= \beta_s^{1,2}(a, b) + \beta_s^{1,2}(ab, c) + \beta_s^1(a, b) \otimes \beta_s^2(ab, c) \\ &\quad - \sum_{\sigma_i \in \mathcal{P}_{[p]}, |\sigma_1| + |\sigma_2| \geq [p]+1} \beta_s^1(a, \sigma_1(b)) \otimes \beta_s^2(a, \sigma_2(b(c-1))) \\ &\quad - \sum_{k=2}^{[p]} \sum_{\sigma_i \in \mathcal{P}_{[p]}, |\sigma_1| + |\sigma_2| \geq [p]+1-k} \beta_s^1(a, \sigma_1(b) \cdot) \otimes \beta_s^2(a, \sigma_2(b) \cdot) \mathcal{I}(\pi_k(c)), \end{aligned} \quad (4.8)$$

where the extra terms are caused by the truncation in (4.6). Then it can be computed that, for $a \in \mathcal{G}_{[p]}$,

$$\begin{aligned} & (\beta_t - \beta_s)(g_t, a) \\ &= \int_0^t \beta_u^1(g_u) dg_u \otimes (\beta_t^2 - \beta_s^2)(g_t, a) \\ &\quad + \left(\int_s^t \beta_u^1(g_u) dg_u - \beta_s^1(g_s, g_{s,t}) \right) \otimes \beta_s^2(g_t, a) + \left(\beta_t^{1,2} - \beta_s^{1,2} \right)(g_t, a) \\ &\quad + \sum_{\sigma_i \in \mathcal{P}_{[p]}, |\sigma_1| + |\sigma_2| \geq [p]+1} \beta_s^1(g_s, \sigma_1(g_{s,t})) \otimes \beta_s^2(g_s, \sigma_2(g_{s,t}(a - \sigma_0(a)))) \\ &\quad + \sum_{k=2}^{[p]} \sum_{\sigma_i \in \mathcal{P}_{[p]}, |\sigma_1| + |\sigma_2| \geq [p]+1-k} \beta_s^1(g_s, \sigma_1(g_{s,t}) \cdot) \otimes \beta_s^2(g_s, \sigma_2(g_{s,t}) \cdot) \mathcal{I}(\pi_k(a)). \end{aligned} \quad (4.9)$$

Using Condition 23, (4.9) can be extended to $T^{([p])}(\mathcal{V})$. Combining with (3.12), (3.13), (4.7) and $\|\sigma(g_{s,t})\| \leq \|g\|_{p-var, [s,t]}^{|\sigma|}$, we have, with $\omega = \omega_1 + \omega_2 + \|g\|_{p-var}^p$ and $\theta = \min(\theta_1, \theta_2)$,

$$\|(\beta_t - \beta_s)(g_t, \cdot)\|_k \leq C_{p,\omega(0,T)} \left\| \beta^1 \right\|_{\theta_1}^{\omega_1} \left\| \beta^2 \right\|_{\theta_2}^{\omega_2} \omega(s, t)^{\theta - \frac{k}{p}}.$$

Similarly we have $\|\beta_s(g_s, \cdot)\| \leq C_{p,\omega(0,T)} \left\| \beta^1 \right\|_{\theta_1}^{\omega_1} \left\| \beta^2 \right\|_{\theta_2}^{\omega_2}$, and the estimate (4.2) holds. ■

4.2 Algebra

Based on Stone-Weierstrass Theorem, being an algebra can be viewed as one of the most important properties of dominated paths. When working with measures on paths space, the algebra structure is compatible with the filtration generated, and the space of previsible integrable one-forms forms an algebra.

Proposition 35 (Algebra) *Let $g \in C^{p-var}([0, T], \mathcal{G}_{[p]})$. Suppose $\int_0^t \beta_u^i(g_u) dg_u : [0, T] \rightarrow \mathcal{U}^i$, $i = 1, 2$, are two dominated paths satisfying $\|\beta^i\|_{\theta_i}^{\omega_i} < \infty$ for control ω_i and $\theta_i > 1$, $i = 1, 2$ (as defined in (3.10)). Then there exists a dominated path $\int_0^t \beta_u(g_u) dg_u : [0, T] \rightarrow \mathcal{U}^1 \otimes \mathcal{U}^2$ such that with control $\omega := \omega_1 + \omega_2 + \|g\|_{p-var}^p$ and $\theta := \min(\theta_1, \theta_2)$,*

$$\|\beta\|_{\theta}^{\omega} \leq C_{p,\omega(0,T)} \left\| \beta^1 \right\|_{\theta_1}^{\omega_1} \left\| \beta^2 \right\|_{\theta_2}^{\omega_2}, \quad (4.10)$$

$$\text{and } \int_0^t \beta_u(g_u) dg_u = \int_0^t \beta_u^1(g_u) dg_u \otimes \int_0^t \beta_u^2(g_u) dg_u, \quad \forall 0 \leq t \leq T.$$

Remark 36 *The fact that dominated paths form an algebra does not necessarily follow from the statement that the iterated integral of two dominated paths is canonically defined. This depends on the definition of the formal integral $\iint_{0 < u_1 < u_2 < T} \delta g_{0,u_1} \otimes \delta g_{0,u_2}$ (see Condition 25 and the discussion thereafter). The reason is that the integration by parts formula may not hold:*

$$\iint_{0 < u_1 < u_2 < T} \delta g_{0,u_1} \otimes \delta g_{0,u_2} + \iint_{0 < u_2 < u_1 < T} \delta g_{0,u_1} \otimes \delta g_{0,u_2} \stackrel{?}{=} g_{0,T} \otimes g_{0,T}, \quad \forall g \in C([0, T], \mathcal{G}). \quad (4.11)$$

When \mathcal{G} is the nilpotent Lie group, (4.11) holds; when \mathcal{G} is the Butcher group, generally (4.11) does not hold.

Proof. We define $\beta \in C([0, T], B(\mathcal{G}_{[p]}, \mathcal{U}^1 \otimes \mathcal{U}^2))$ by, for $a, b \in \mathcal{G}_{[p]}$ and $s \in [0, T]$,

$$\begin{aligned} \beta_s(a, b) &= \beta_s^1(g_s, g_s^{-1}a(b-1)) \otimes \int_0^s \beta_u^2(g_u) dg_u + \int_0^s \beta_u^1(g_u) dg_u \otimes \beta_s^2(g_s, g_s^{-1}a(b-1)) \\ &+ \sum_{\sigma_i \in \mathcal{P}_{[p]}, |\sigma_1|+|\sigma_2| \leq [p]} \beta_s^1(g_s, \cdot) \otimes \beta_s^2(g_s, \cdot) (\sigma_1 * \sigma_2)(g_s^{-1}a(b-1)), \end{aligned} \quad (4.12)$$

where $\sigma_1 * \sigma_2 \in L(\mathcal{V}^{\otimes(|\sigma_1|+|\sigma_2|)}, \mathcal{V}^{\otimes|\sigma_1|} \otimes \mathcal{V}^{\otimes|\sigma_2|})$ is defined in Condition 24 and $\beta_s^1(g_s, \cdot) \otimes \beta_s^2(g_s, \cdot)$ denotes the unique continuous linear mapping from $T^{([p])}(\mathcal{V})^{\otimes 2}$ to $\mathcal{U}^1 \otimes \mathcal{U}^2$ satisfying

$$\beta_s^1(g_s, \cdot) \otimes \beta_s^2(g_s, \cdot)(v_1 \otimes v_2) = \beta_s^1(g_s, v_1) \otimes \beta_s^2(g_s, v_2), \forall v_1, v_2 \in T^{([p])}(\mathcal{V}).$$

Since there exists a unique multiplicative function associated with an almost multiplicative function [25], we have

$$\int_0^t \beta_u^1(g_u) dg_u \otimes \int_0^t \beta_u^2(g_u) dg_u = \int_0^t \beta_u(g_u) dg_u, \forall t \in [0, T]. \quad (4.13)$$

When proving that β satisfies the slowly-varying condition, there is an extra term caused by truncation up to level $[p]$ in (4.12). The extra term can be derived by using that for $\tau_i, \sigma_i \in \mathcal{P}_{[p]}$ and $a, b \in \mathcal{G}_{[p]}$, $(\tau_1 \otimes \tau_2)$ denotes the continuous linear mapping that satisfies $(\tau_1 \otimes \tau_2)(a) := \tau_1(a) \otimes \tau_2(a)$, $\forall a \in \mathcal{G}_{[p]}$, and $(a_1 \otimes a_2)(b_1 \otimes b_2) := (a_1 b_1) \otimes (a_2 b_2)$

$$\begin{aligned} &(\tau_1 \otimes \tau_2)(a(b-1)) \\ &= \sum_{\sigma_i \in \mathcal{P}_{[p]}} c(\tau_1, \rho_1, \sigma_1) c(\tau_2, \rho_2, \sigma_2) (\rho_1(a) \otimes \rho_2(a)) ((\sigma_1 \otimes \sigma_2)(b-1)), \\ & \quad (a \otimes a) ((\sigma_1 \otimes \sigma_2)(b-1)) \\ &= \sum_{\tau_i \in \mathcal{P}_{[p]}} c(\tau_1, \rho_1, \sigma_1) c(\tau_2, \rho_2, \sigma_2) (\rho_1(a) \otimes \rho_2(a)) ((\sigma_1 \otimes \sigma_2)(b-1)), \end{aligned}$$

where the constant $c(\tau_i, \rho_i, \sigma_i)$, $i = 1, 2$, denotes the number of $\rho_i \otimes \sigma_i$ in the comultiplication of τ_i . Hence

$$\begin{aligned} &\sum_{|\tau_1|+|\tau_2| \geq [p]+1} (\tau_1 \otimes \tau_2)(a(b-1)) - \sum_{|\sigma_1|+|\sigma_2| \geq [p]+1} (a \otimes a) ((\sigma_1 \otimes \sigma_2)(b-1)) \\ &= \sum_{|\tau_1|+|\tau_2| \geq [p]+1, |\sigma_1|+|\sigma_2| \leq [p]} c(\tau_1, \rho_1, \sigma_1) c(\tau_2, \rho_2, \sigma_2) (\rho_1(a) \otimes \rho_2(a)) ((\sigma_1 \otimes \sigma_2)(b-1)). \end{aligned}$$

Then, for $s < t$ and $a \in \mathcal{G}_{[p]}$, it can be computed that

$$\begin{aligned} &(\beta_t - \beta_s)(g_t, a) \\ &= (\beta_t^1 - \beta_s^1)(g_t, a) \otimes \int_0^s \beta_v^2(g_v) dg_v + \int_0^s \beta_v^1(g_v) dg_v \otimes (\beta_t^2 - \beta_s^2)(g_t, a) \\ &+ \beta_t^1(g_t, a) \otimes \left(\int_s^t \beta_v^2(g_v) dg_v - \beta_s^2(g_s, g_{s,t}) \right) + \left(\int_s^t \beta_v^1(g_v) dg_v - \beta_s^1(g_s, g_{s,t}) \right) \otimes \beta_t^2(g_t, a) \\ &+ (\beta_t^1 - \beta_s^1)(g_t, a) \otimes \beta_s^2(g_s, g_{s,t}) + \beta_s^1(g_s, g_{s,t}) \otimes (\beta_t^2 - \beta_s^2)(g_t, a) \\ &+ \sum_{|\sigma_1|+|\sigma_2| \leq [p]} (\beta_t^1(g_t, \cdot) \otimes \beta_t^2(g_t, \cdot) - \beta_s^1(g_t, \cdot) \otimes \beta_s^2(g_t, \cdot)) ((\sigma_1 \otimes \sigma_2)(a - \sigma_0(a))) \\ &+ \sum_{\substack{|\tau_1|+|\tau_2| \geq [p]+1 \\ |\sigma_1|+|\sigma_2| \leq [p]}} c(\tau_1, \rho_1, \sigma_1) c(\tau_2, \rho_2, \sigma_2) \beta_s^1(g_s, \rho_1(g_{s,t}, \cdot)) \otimes \beta_s^2(g_s, \rho_2(g_{s,t}, \cdot)) ((\sigma_1 \otimes \sigma_2)(a - \sigma_0(a))). \end{aligned} \quad (4.14)$$

We represent $\sigma_1 \otimes \sigma_2$ as a continuous linear mapping $\sigma_1 * \sigma_2$ by using Condition 24. Similar as in the proof of Proposition 32, we have (4.10) holds based on Condition 23, (4.14), (3.12), (3.13) and $\|\rho(g_{s,t})\| \leq \|g\|_{p-var, [s,t]}^{[p]}$. ■

Remark 37 In (4.12), β is defined as the sum of three time-varying cocyclic one-forms (denoted by $(\eta^i)_{i=1,2,3}$). Although β is slowly-varying as we proved above, $(\eta^i)_{i=1,2,3}$ are generally not slowly-varying. Roughly speaking, the difference between η_t^i and η_s^i for $s < t$ would be comparable to $\|g\|_{p-var, [s,t]}$ that is not (slow) enough to integrate against g when $p \geq 2$.

Remark 38 Proposition 35 is a special (important) case of Proposition 39.

4.3 Composition

For $\gamma > 0$, let $\lfloor \gamma \rfloor$ denote the largest integer which is strictly less than γ . Let \mathcal{U} and \mathcal{W} be two Banach spaces. We say $f \in C^\gamma(\mathcal{U}, \mathcal{W})$, if $f : \mathcal{U} \rightarrow \mathcal{W}$ is $\lfloor \gamma \rfloor$ -times Fréchet differentiable and

$$\|f\|_{\text{Lip}(\gamma), R} := \sup_{x \neq y, \|x\| \vee \|y\| \leq R} \frac{\|(D^{\lfloor \gamma \rfloor} f)(x) - (D^{\lfloor \gamma \rfloor} f)(y)\|}{\|x - y\|^{\gamma - \lfloor \gamma \rfloor}} \leq C_R, \forall R > 0. \quad (4.15)$$

Proposition 39 (Composition) Let $g \in C^{p-var}([0, T], \mathcal{G}_{[p]})$, and suppose $X := \int_0^\cdot \beta_u(g_u) dg_u : [0, T] \rightarrow \mathcal{U}$ is a dominated path satisfying $\|\beta\|_\omega^\theta < \infty$ for control ω and $\theta > 1$. For Banach space \mathcal{W} and $f \in C^\gamma(\mathcal{U}, \mathcal{W})$, $\gamma > p$, there exists a dominated path $\int_0^\cdot \hat{\beta}(g) dg : [0, T] \rightarrow \mathcal{W}$ such that with control $\hat{\omega} = \omega + \|g\|_{p-var}^p$ and $\hat{\theta} := \min\left(\theta, \frac{\gamma}{p}, \frac{[p]+1}{p}\right)$,

$$\left\| \hat{\beta} \right\|_{\hat{\theta}}^{\hat{\omega}} \leq C_{p, \hat{\omega}(0, T)} \|f\|_{\text{Lip}(\gamma), \|X\|_\infty} \max\left(\|\beta\|_\theta^\omega, (\|\beta\|_\theta^\omega)^{[p]}\right) \quad (4.16)$$

where $\|X\|_\infty := \sup_{t \in [0, T]} \|X_t\|$, and

$$\int_0^t \hat{\beta}_u(g_u) dg_u = f(X_t) - f(0), \forall 0 \leq t \leq T.$$

Proof. We rescale f by $\|f\|_{\text{Lip}(\gamma), \|X\|_\infty}^{-1}$ and assume $\|f\|_{\text{Lip}(\gamma), \|X\|_\infty} = 1$.

Define $\hat{\beta} \in C([0, T], B(\mathcal{G}_{[p]}, \mathcal{W}))$ by, for $a, b \in \mathcal{G}_{[p]}$ and $s \in [0, T]$,

$$\hat{\beta}_s(a, b) := \sum_{l=1}^{[p]} \frac{1}{l!} (D^l f)(X_s) \sum_{\sigma_i \in \mathcal{P}_{[p]}, |\sigma_1| + \dots + |\sigma_l| \leq [p]} \beta_s(g_s, \cdot)^{\otimes l} (\sigma_1 * \dots * \sigma_l)(g_s^{-1} a (b - 1)), \quad (4.17)$$

where $\sigma_1 * \dots * \sigma_l \in L(\mathcal{V}^{\otimes(|\sigma_1| + \dots + |\sigma_l|)}, \mathcal{V}^{\otimes|\sigma_1|} \otimes \dots \otimes \mathcal{V}^{\otimes|\sigma_l|})$ is defined in Condition 24 and $\beta_s(g_s, \cdot)^{\otimes l}$ denotes the unique continuous linear mapping from $T^{([p])}(\mathcal{V})^{\otimes l}$ to $\mathcal{U}^{\otimes l}$ satisfying

$$\beta_s(g_s, \cdot)^{\otimes l} (v_1 \otimes \dots \otimes v_l) = \beta_s(g_s, v_1) \otimes \dots \otimes \beta_s(g_s, v_l), \quad \forall v_i \in T^{([p])}(\mathcal{V}), \quad i = 1, \dots, l.$$

Since the multiplicative function associated with an almost multiplicative function is unique [25], we have $f(X_t) = f(0) + \int_0^t \hat{\beta}_u(g_u) dg_u$, $\forall t \in [0, T]$. Then we check that $\hat{\beta}$ satisfies the slowly-varying condition. As in the proof of Proposition 35, the extra term caused by truncation up to level $[p]$ in (4.17) can be estimated based on the formula that for $\tau_i, \sigma_i \in \mathcal{P}_{[p]}$ and $a, b \in \mathcal{G}_{[p]}$,

$$\begin{aligned} & \sum_{|\tau_1| + \dots + |\tau_l| \geq [p]+1} (\tau_1 \otimes \dots \otimes \tau_l)(a(b-1)) - \sum_{|\sigma_1| + \dots + |\sigma_l| \geq [p]+1} a^{\otimes l} ((\sigma_1 \otimes \dots \otimes \sigma_l)(b-1)) \\ &= \sum_{\substack{|\tau_1| + \dots + |\tau_l| \geq [p]+1 \\ |\sigma_1| + \dots + |\sigma_l| \leq [p]}} c(\tau_1, \rho_1, \sigma_1) \dots c(\tau_l, \rho_l, \sigma_l) (\rho_1(a) \otimes \dots \otimes \rho_l(a)) ((\sigma_1 \otimes \dots \otimes \sigma_l)(b-1)), \end{aligned}$$

where $c(\tau_i, \rho_i, \sigma_i)$ denotes the number of $\rho_i \otimes \sigma_i$ in the coproduct of τ_i .

Hence, for $s < t$ and $a \in \mathcal{G}_{[p]}$, it can be computed that

$$\begin{aligned} & (\hat{\beta}_t - \hat{\beta}_s)(g_t, a) \\ &= \sum_{l=1}^{[p]} \frac{1}{l!} \left((D^l f)(X_t) - \sum_{j=0}^{[p]-l} \frac{1}{j!} (D^{l+j} f)(X_s) (X_t - X_s)^{\otimes j} \right) \beta_t(g_t, \cdot)^{\otimes l} L^l(a) \\ & \quad + \sum_{l=1}^{[p]} \sum_{j=0}^{[p]-l} \frac{1}{l!} \frac{1}{j!} (D^{l+j} f)(X_s) \left((X_t - X_s)^{\otimes j} - (\beta_s(g_s, g_{s,t}))^{\otimes j} \right) \beta_t(g_t, \cdot)^{\otimes l} L^l(a) \\ & \quad + \sum_{l=1}^{[p]} \frac{1}{l!} (D^l f)(X_s) \left((\beta_s(g_s, g_{s,t}) + \beta_t(g_t, \cdot))^{\otimes l} - (\beta_s(g_s, g_{s,t}) + \beta_s(g_t, \cdot))^{\otimes l} \right) L^l(a) \\ & \quad + \sum_{l=1}^{[p]} \frac{1}{l!} (D^l f)(X_s) \beta_s(g_s, \cdot)^{\otimes l} R^l(a), \end{aligned} \quad (4.18)$$

where

$$\begin{aligned} L^l(a) & : = \sum_{|\sigma_1| + \dots + |\sigma_l| \leq [p]} (\sigma_1 \otimes \dots \otimes \sigma_l)(a), \\ R^l(a) & : = \sum_{\substack{|\tau_1| + \dots + |\tau_l| \geq [p]+1 \\ |\sigma_1| + \dots + |\sigma_l| \leq [p]}} c(\tau_1, \rho_1, \sigma_1) \dots c(\tau_l, \rho_l, \sigma_l) (\rho_1(g_{s,t}) \otimes \dots \otimes \rho_l(g_{s,t})) ((\sigma_1 \otimes \dots \otimes \sigma_l)(a-1)). \end{aligned}$$

Based on Condition 23 and Condition 24, we represent $L^l(\cdot)$ and $R^l(\cdot)$ as continuous linear mappings on $T^{([p])}(\mathcal{V})$ and extend (4.18) from $\mathcal{G}_{[p]}$ to $T^{([p])}(\mathcal{V})$. By using Taylor's Theorem, estimate (3.12), (3.13), the slow-varying property of β , and that $\|\rho(g)\| \leq \|g\|_{p-var, [s, t]}^{[p]}$, we have (4.16) holds. ■

4.4 Transitivity

Let \mathcal{U} be a Banach space. Suppose the multiplication in the Banach algebra $T^{([p])}(\mathcal{U})$ is defined by (with π_k denotes the projection to $\mathcal{U}^{\otimes k}$) $\pi_k(ab) = \sum_{i=0}^k \pi_i(a) \otimes \pi_{k-i}(b)$ for $k = 0, 1, \dots, [p]$ and $a, b \in T^{([p])}(\mathcal{U})$. Let $1 \oplus \mathcal{U} \oplus \dots \oplus \mathcal{U}^{\otimes [p]}$ denote the closed topological group in $T^{([p])}(\mathcal{U})$ defined by

$$1 \oplus \mathcal{U} \oplus \dots \oplus \mathcal{U}^{\otimes [p]} := \left\{ a \in T^{([p])}(\mathcal{U}) \mid \pi_0(a) = 1 \right\}. \quad (4.19)$$

Proposition 40 (Transitivity) *Let $\gamma := \int_0^\cdot \beta_u(g_u) dg_u : [0, T] \rightarrow \mathcal{U}$ be a path dominated by g satisfying $\|\beta\|_\theta^\omega < \infty$ for control ω and $\theta > 1$. Define $\Gamma : [0, T] \rightarrow 1 \oplus \mathcal{U} \oplus \dots \oplus \mathcal{U}^{\otimes [p]}$ by*

$$\Gamma_t := 1 + \sum_{n=1}^{[p]} x_t^n \text{ with } x_t^1 := \gamma_t - \gamma_0 \text{ and } x_t^n := \int_0^t x_u^{n-1} \otimes d\gamma_u, \quad n = 2, \dots, [p]$$

where the integrals are defined as in Proposition 32.

For Banach space \mathcal{W} , suppose $\int_0^\cdot \zeta_u(\Gamma_u) d\Gamma_u : [0, T] \rightarrow \mathcal{W}$ is a path dominated by Γ satisfying $\|\zeta\|_\kappa^\rho < \infty$ for control ρ and $\kappa > 1$. Then there exists a dominated path $\int_0^\cdot \tilde{\beta}(g) dg : [0, T] \rightarrow \mathcal{W}$ such that with control $\tilde{\omega} := \omega + \rho + \|g\|_{p-var}^p$ and $\tilde{\theta} := \min(\theta, \kappa) > 1$,

$$\left\| \tilde{\beta} \right\|_{\tilde{\theta}}^{\tilde{\omega}} \leq C_{p, \tilde{\omega}(0, T)} \|\zeta\|_\kappa^\rho \max \left(\|\beta\|_\theta^\omega, (\|\beta\|_\theta^\omega)^{[p]} \right), \quad (4.20)$$

$$\text{and } \int_0^t \tilde{\beta}(g) dg = \int_0^t \zeta_u(\Gamma_u) d\Gamma_u, \quad \forall 0 \leq t \leq T.$$

Remark 41 Whether Γ takes values in the step- $[p]$ nilpotent Lie group or not will depend on the mapping \mathcal{I} in Condition 25.

Remark 42 We can also define $\tilde{\Gamma}$, taking values in the ‘infinite-dimensional’ Butcher group over \mathcal{U} , by

$$\tilde{\Gamma}_t := 1 + \sum_{\sigma \in \mathcal{Q}_{[p]}} x_t^\sigma \text{ with } x_t^\bullet := \gamma_t - \gamma_0, \quad x_t^{\sigma_1 \sigma_2} := x_t^{\sigma_1} \otimes x_t^{\sigma_2}, \quad x_t^{[\sigma_1]} := \int_0^t x_u^{\sigma_1} \otimes d\gamma_u, \quad \forall \sigma_1, \sigma_2 \in \mathcal{Q}_{[p]},$$

where $\mathcal{Q}_{[p]}$ denotes the set of unlabelled ordered forests of degree less or equal to $[p]$ and $[\sigma_1]$ denotes the tree obtained by attaching the forest σ_1 to a new root. $\tilde{\Gamma}$ is well-defined because the set of dominated paths is stable under multiplication and iterated integration.

Proof. We extend the definition of Γ to $\{(s, t) \mid 0 \leq s \leq t \leq T\}$ to make the proof work.

$$\Gamma_{s,t} := 1 + \sum_{n=1}^{[p]} x_{s,t}^n \text{ with } x_{s,t}^1 := \gamma_t - \gamma_s \text{ and } x_{s,t}^n := \int_s^t x_{s,u}^{n-1} \otimes d\gamma_u, \quad n = 2, \dots, [p]. \quad (4.21)$$

We view $\mathcal{U} \oplus \dots \oplus \mathcal{U}^{\otimes [p]}$ as a Banach space (with norm $\sum_{k=1}^{[p]} \|\pi_k(\cdot)\|$), and define $B_{s,t} \in B(\mathcal{G}_{[p]}, \mathcal{U} \oplus \dots \oplus \mathcal{U}^{\otimes [p]})$ for $s < t$ by

$$B_{s,t} := \sum_{n=1}^{[p]} \beta_{s,t}^n, \quad (4.22)$$

where $\beta_{s,t}^n \in B(\mathcal{G}_{[p]}, \mathcal{U}^{\otimes n})$ are defined by, for $a, b \in \mathcal{G}_{[p]}$, (with \mathcal{I} in Condition 25)

$$\begin{aligned} \beta_{s,t}^1(a, b) &: = \beta_t(a, b), \\ \beta_{s,t}^{n+1}(a, b) &: = x_{s,t}^n \otimes \beta_t(g_t, g_t^{-1}a(b-1)) + \beta_{s,t}^n(g_t, \cdot) \otimes \beta_t(g_t, \cdot) \mathcal{I}(g_t^{-1}a(b-1)), \quad n \geq 1. \end{aligned} \quad (4.23)$$

Based on Proposition 32 and Γ defined in (4.21), it can be proved inductively that $\|\beta_{s,\cdot}^n\|_\theta^{\hat{\omega}} \leq C_{p, \hat{\omega}(0, T)} (\|\beta\|_\theta^\omega)^n$ for $n = 1, \dots, [p]$ with $\hat{\omega} = \omega + \|g\|_{p-var}^p$ for any $s \in [0, T]$. Hence,

$$\|B_{s,\cdot}\|_\theta^{\hat{\omega}} \leq C_{p, \hat{\omega}(0, T)} \max \left(\|\beta\|_\theta^\omega, (\|\beta\|_\theta^\omega)^{[p]} \right), \quad \forall s \in [0, T], \quad (4.24)$$

$$\Gamma_{s,t} = 1 + \int_s^t B_{s,u}(g_u) dg_u, \quad \|\Gamma_{s,t} - 1 - B_{s,s}(g_s, g_{s,t})\| \leq \|B_{s,\cdot}\|_\theta^{\hat{\omega}} \hat{\omega}(s, t)^\theta, \quad \forall 0 \leq s \leq t \leq T. \quad (4.25)$$

Then we prove a simple property of $B_{t,t}$ that for $k = 1, \dots, [p]$,

$$B_{t,t}(g_t, v) \in \mathcal{U} \oplus \dots \oplus \mathcal{U}^{\otimes k}, \quad \forall v \in \mathcal{V}^{\otimes k}, \quad \forall t. \quad (4.26)$$

Equivalently, we prove that, for $n = 1, \dots, [p]$,

$$\beta_{t,t}^n(g_t, v) = 0, \quad \forall v \in \mathcal{V}^{\otimes k}, \quad k = 0, 1, \dots, n-1, \quad \forall t, \quad (4.27)$$

which holds for $n = 1$. Suppose (4.27) holds for some $n = 1, \dots, [p] - 1$. Then, by using the property in Condition 25 that $\mathcal{I}(\mathcal{V}^{\otimes k}) \subseteq \sum_{j_1 \geq 1, j_1 + j_2 = k} \mathcal{V}^{\otimes j_1} \otimes \mathcal{V}^{\otimes j_2}$ and by using the inductive hypothesis, we have that, if $\beta_{t,t}^{n+1}(g_t, v) = \beta_{t,t}^n(g_t, \cdot) \otimes \beta_t(g_t, \cdot) \mathcal{I}(v) \neq 0$ for some $v \in \mathcal{V}^{\otimes k}$, then $k = j_1 + j_2$ for some $j_1 \geq n$ and $j_2 \geq 1$, which implies $k \geq n + 1$ and the induction is complete.

Based on the definition of $\beta_{s,t}^{n+1}$ in (4.23), it can be proved by induction that

$$\beta_{s,t}^n(g_t, \cdot) = \sum_{i=1}^n x_{s,t}^{n-i} \otimes \beta_{s,t}^i(g_t, \cdot), \quad \forall v \in T^{([p])}(\mathcal{V}). \quad (4.28)$$

Hence the relationship holds:

$$B_{s,t}(g_t, \cdot) = \Gamma_{s,t} B_{t,t}(g_t, \cdot), \quad \forall 0 \leq s \leq t \leq T, \quad (4.29)$$

where the multiplication between $\Gamma_{s,t}$ and $B_{t,t}(g_t, \cdot)$ is in the algebra $T^{([p])}(\mathcal{U})$.

Then we prove

$$\Gamma_{0,s} \Gamma_{s,t} = \Gamma_{0,t}, \quad \forall 0 \leq s \leq t \leq T. \quad (4.30)$$

Equivalently, we prove,

$$x_{0,t}^n = \sum_{i=0}^n x_{0,s}^{n-i} \otimes x_{s,t}^i, \quad n = 1, \dots, [p], \quad 0 \leq s \leq t \leq T, \quad (4.31)$$

which holds when $n = 1$. Suppose (4.31) holds for some $n = 1, \dots, [p] - 1$. Based on (4.28) and the inductive hypothesis (4.31), we have

$$\beta_{0,t}^n(g_t, \cdot) = \sum_{l=1}^n x_{0,s}^{n-l} \otimes \beta_{s,t}^l(g_t, \cdot), \quad \forall 0 \leq s \leq t \leq T. \quad (4.32)$$

Then (4.31) holds for $n + 1$ based on the definition of the integral in Proposition 32, the inductive hypothesis (4.31) and (4.32).

Then we prove that, if a path is dominated by Γ , then it is dominated by g . Suppose $\int_0^\cdot \zeta(\Gamma_{0,u}) d\Gamma_{0,u} : [0, T] \rightarrow \mathcal{W}$ is a path dominated by $t \mapsto \Gamma_{0,t}$. Define $\tilde{\beta} : [0, T] \rightarrow B(\mathcal{G}_{[p]}, \mathcal{W})$ by

$$\tilde{\beta}_s(a, b) = \zeta_s(\Gamma_{0,s}, B_{s,s}(g_s, g_s^{-1}a(b-1))), \quad \forall a, b \in \mathcal{G}_{[p]}.$$

That $\int_0^\cdot \tilde{\beta}_u(g_u) dg_u = \int_0^\cdot \zeta_u(\Gamma_{0,u}) d\Gamma_{0,u}$ follows from the uniqueness of the multiplicative function and (4.25). Since $B_{t,t}(g_t, \cdot)$ takes values in $\mathcal{U} \oplus \dots \oplus \mathcal{U}^{\otimes [p]}$, by using (4.1), (4.29) and (4.30), we have $\zeta_s(\Gamma_{0,t}, B_{t,t}(g_t, \cdot)) = \zeta_s(\Gamma_{0,s} \Gamma_{s,t}, B_{t,t}(g_t, \cdot)) = \zeta_s(\Gamma_{0,s}, \Gamma_{s,t} B_{t,t}(g_t, \cdot)) = \zeta_s(\Gamma_{0,s}, B_{s,t}(g_t, \cdot))$. Hence,

$$\begin{aligned} & (\tilde{\beta}_t - \tilde{\beta}_s)(g_t, \cdot) \\ &= (\zeta_t - \zeta_s)(\Gamma_{0,t}, B_{t,t}(g_t, \cdot)) + \zeta_s(\Gamma_{0,t}, B_{t,t}(g_t, \cdot)) - \zeta_s(\Gamma_{0,s}, B_{s,s}(g_t, \cdot)) \\ &= (\zeta_t - \zeta_s)(\Gamma_{0,t}, B_{t,t}(g_t, \cdot)) + \zeta_s(\Gamma_{0,s}, (B_{s,t} - B_{s,s})(g_t, \cdot)). \end{aligned}$$

Then based on (4.26), the slowly-varying property of $t \mapsto \zeta_t$ and $t \mapsto B_{s,t}$, and (4.24), we have (4.20) holds. ■

5 Rough integration

Recall that $L(\mathcal{V}, \mathcal{U})$ denotes the set of continuous linear mappings from \mathcal{V} to \mathcal{U} , and $L(\mathcal{V}, \mathcal{U})$ becomes a Banach space when equipped with the operator norm. Recall the group $1 \oplus \mathcal{U} \oplus \dots \oplus \mathcal{U}^{\otimes [p]}$ defined in (4.19).

Corollary 43 (Rough Integration) *Let $\mathcal{G}_{[p]}$ be the step- $[p]$ nilpotent Lie group over the Banach space \mathcal{V} . For $\gamma > p - 1$ and Banach space \mathcal{U} , suppose $f \in C^\gamma(\mathcal{V}, L(\mathcal{V}, \mathcal{U}))$ (as defined in (4.15)). For $g \in C^{p-var}([0, T], \mathcal{G}_{[p]})$, we define $\beta \in ([0, T], B(\mathcal{G}_{[p]}, \mathcal{U}))$ by (with $x_s := \pi_1(g_s)$ and π_l denotes the projection to $\mathcal{V}^{\otimes l}$)*

$$\beta_s(a, b) := \sum_{l=0}^{[p]-1} \frac{1}{l!} (D^l f)(x_s) \pi_{l+1}(g_s^{-1}a(b-1)), \quad \forall a, b \in \mathcal{G}_{[p]}, \quad \forall s \in [0, T]. \quad (5.1)$$

Then $\int_0^\cdot \beta_u(g_u) dg_u$ is a dominated path. In addition, by using the mapping \mathcal{I} in (3.5) and the integration in Proposition 32, we define $Y : [0, T] \rightarrow 1 \oplus \mathcal{U} \oplus \dots \oplus \mathcal{U}^{\otimes [p]}$ by

$$Y_t = 1 + \sum_{n=1}^{[p]} y_t^n \quad \text{with } y_t^1 := \int_0^t \beta_u(g_u) dg_u \text{ and } y_t^n := \int_0^t y_u^{n-1} \otimes dy_u^1, \quad n = 2, \dots, [p], \quad \forall t \in [0, T].$$

Then Y coincides with the rough integral in [25].

Remark 44 Let $\gamma > p - 1$. Suppose $F : [0, T] \rightarrow C^\gamma(\mathcal{V}, L(\mathcal{V}, \mathcal{U}))$ is a time-varying Lipschitz one-form, satisfying, for some control ω and $\theta > 1$,

$$\|((D^l F_t) - (D^l F_s))(x_t)\| \leq \omega(s, t)^{\theta - \frac{l+1}{p}}, \quad \forall 0 \leq s \leq t \leq T, \quad l = 0, 1, \dots, [p] - 1. \quad (5.2)$$

If we modify the definition of β_s in (5.1) by replacing f with F_s , then it can be proved similarly that $\int_0^\cdot \beta_u(g_u) dg_u$ is a dominated path.

Remark 45 Let X be a p -rough path, H be a q -rough path, $p^{-1} + q^{-1} > 1$, $p \geq q$, and let $(x, h) \mapsto \alpha(x, h)$ be a Lipschitz function that is $\text{Lip}(\gamma)$ in x and $\text{Lip}(\kappa)$ in h . Then the rough integral $\int \alpha(X, H) dX$ is well defined when $\gamma > p - 1$ and $(\kappa \wedge 1)q^{-1} + p^{-1} > 1$, because $t \mapsto \alpha(\cdot, h_t)$ is a time-varying Lipschitz function satisfying (5.2).

Remark 46 The integration in Corollary 43 is a generalization of Lyons' original integration [25] in the sense that g can be a weakly geometric rough path (see also [3]).

Remark 47 When g takes values in Butcher group, the one-form can be defined by,

$$\beta_s(a, b) := \sum_{l=0}^{[p]-1} \frac{1}{l!} (D^l f)(x_s) \sigma_{l+1}(g_s^{-1}a(b-1)), \quad \forall a, b \in \mathcal{G}_{[p]}, \quad \forall s \in [0, T],$$

where $\sigma_{l+1} \in \mathcal{P}_{[p]}$ denotes the unlabelled tree obtained by attaching l branches with one node to a new root, e.g. $\sigma_1 = \bullet$, $\sigma_2 = [\bullet]$, $\sigma_3 = [\bullet\bullet]$. Then $\int \beta(g) dg$ is a dominated path and coincides with the integral in [15].

Proof. Since $D^l f \in C^{\gamma-l}(\mathcal{V}, L(\mathcal{V}^{\otimes l}, L(\mathcal{V}, \mathcal{U})))$ is symmetric in $\mathcal{V}^{\otimes l}$ and the projection of $\pi_i(a)$, $a \in \mathcal{G}_{[p]}$, to the space of symmetric tensors is $(i!)^{-1}(\pi_1(a))^{\otimes i}$ (see [25]), it can be computed that, for $a, b \in \mathcal{G}_{[p]}$,

$$\beta_s(a, b) = \sum_{l=0}^{[p]-1} \left(\sum_{j=0}^{[p]-1-l} (D^{l+j} f)(x_s) \frac{(\pi_1(a) - x_s)^{\otimes j}}{j!} \right) \otimes \pi_{l+1}(b) \quad (\text{since } \pi_0(b) = 1). \quad (5.3)$$

Then we check that β satisfies the slowly-varying condition. Since $\beta_s(a, \cdot)$ is a continuous linear mapping, based on Condition 23, the equality (5.3) holds when b is replaced by $v \in \mathcal{V}^{\otimes k}$, $k = 1, \dots, [p]$. Then, for $0 \leq s \leq t \leq T$ and $k = 1, 2, \dots, [p]$, we have

$$(\beta_t - \beta_s)(g_t, v) = \left((D^{k-1} f)(x_t) - \sum_{j=0}^{[p]-k} (D^{k+j-1} f)(x_s) \frac{(x_t - x_s)^{\otimes j}}{j!} \right) \otimes v, \quad \forall v \in \mathcal{V}^{\otimes k}. \quad (5.4)$$

Since $f \in C^\gamma(\mathcal{V}, L(\mathcal{V}, \mathcal{U}))$ for $\gamma > p - 1$, by using (5.4) and Taylor's theorem, we have

$$\|(\beta_t - \beta_s)(g_t, \cdot)\|_k \leq C \|x_t - x_s\|^{\gamma-k+1} \leq C (\|g\|_{p\text{-var}, [s, t]}^p)^{\frac{\gamma+1}{p} - \frac{k}{p}}.$$

Since $\{D^k f\}_{k=0}^{[p]-1}$ are bounded on bounded sets, we have $\sup_{s \in [0, T]} \|\beta_s(g_s, \cdot)\| < \infty$ and β satisfies the slowly-varying condition.

Then, by working on the local expansion of Y , we check that Y coincides with the rough integral in [25].

For $0 \leq s \leq t \leq T$, we denote

$$Y_{s,t} := Y_s^{-1} Y_t \quad \text{and} \quad y_{s,t}^n := \pi_n(Y_{s,t}), \quad n = 0, 1, \dots, [p].$$

We define $H_s \in B(\mathcal{G}_{[p]}, \mathcal{U} \oplus \dots \oplus \mathcal{U}^{\otimes [p]})$ for $s \in [0, T]$ by

$$H_s = \sum_{n=1}^{[p]} \eta_s^n,$$

where $\eta_s^n \in B(\mathcal{G}_{[p]}, \mathcal{U}^{\otimes n})$ are defined by

$$\eta_s^1(a, b) := \beta_s(a, b), \quad \eta_s^n(a, b) := \eta_s^{n-1}(g_s, \cdot) \otimes \eta_s^1(g_s, \cdot) \mathcal{I}(g_s^{-1}a(b-1)), \quad \forall a, b \in \mathcal{G}_{[p]}.$$

Hence $H_s = B_{s,s}$ for $B_{s,s}$ defined at (4.22) and $\eta_s^n = \beta_{s,s}^n$ for $\beta_{s,s}^n$ defined at (4.23). Then based on (4.25), there exist control ω and $\theta > 1$, s.t.

$$\|Y_{s,t} - 1 - H_s(g_s, g_{s,t})\| \leq \omega(s, t)^\theta, \quad \forall 0 \leq s \leq t \leq T. \quad (5.5)$$

To write $H_s(g_s, g_{s,t})$ in a more explicit form, we define the mappings $\mathcal{I}^n \in L(T^{([p])}(\mathcal{V}), (T^{([p])}(\mathcal{V}))^{\otimes(n+1)})$, $n = 1, \dots, [p] - 1$, by (with \mathcal{I} in (3.5) and I_d denotes the identity function on $T^{([p])}(\mathcal{V})$):

$$\mathcal{I}^1 := \mathcal{I}, \quad \mathcal{I}^n = (\mathcal{I}^{n-1} \otimes I_d) \circ \mathcal{I}^1, \quad n = 2, \dots, [p] - 1. \quad (5.6)$$

Then, it can be proved inductively that

$$\eta_s^1(g_s, g_{s,t}) = \beta_s(g_s, g_{s,t}) \quad \text{and} \quad \eta_s^n(g_s, g_{s,t}) = \beta_s(g_s, \cdot)^{\otimes n} \mathcal{I}^{n-1}(g_{s,t}), \quad n = 2, \dots, [p].$$

Combined with (5.5), we have that, there exist control ω and $\theta > 1$ such that for any $0 \leq s \leq t \leq T$,

$$\|y_{s,t}^1 - \beta_s(g_s, g_{s,t})\| \leq \omega(s, t)^\theta, \quad \|y_{s,t}^n - \beta_s(g_s, \cdot)^{\otimes n} \mathcal{I}^{n-1}(g_{s,t})\| \leq \omega(s, t)^\theta, \quad n = 2, \dots, [p]. \quad (5.7)$$

Then we check that

$$\mathcal{I}^n(a) = \sum_{j_1 + \dots + j_{n+1} \leq [p], j_i \geq 1} \sum_{\rho \in OS(j_1, \dots, j_{n+1})} \rho^{-1}(\pi_{j_1 + \dots + j_{n+1}}(a)), \quad n = 1, \dots, [p] - 1, \quad \forall a \in \mathcal{G}_{[p]}, \quad (5.8)$$

where $OS(j_1, \dots, j_{n+1})$ denotes the ordered shuffles (p73 [26]). Indeed, we first suppose a is an element in the step- $[p]$ nilpotent Lie group over a finite dimensional subspace of \mathcal{V} . Then based on Chow-Rashevskii Connectivity Theorem, there exists some continuous bounded variation path x , which takes values in the finite dimensional subspace of \mathcal{V} and satisfies $S_{[p]}(x)_{0,1} = a$. By comparing the expression on the r.h.s. of (5.8) with the expression (4.9) on p74 in [26], we can rewrite (5.8) in the form (with $1_{[p],n+1}$ denotes the projection of $(T^{([p])}(\mathcal{V}))^{\otimes(n+1)}$ to $\sum_{j_1 + \dots + j_{n+1} \leq [p], j_i \geq 1} \mathcal{V}^{\otimes j_1} \otimes \dots \otimes \mathcal{V}^{\otimes j_{n+1}}$)

$$\mathcal{I}^n(a) = 1_{[p],n+1} \left(\int \dots \int_{0 < u_1 < \dots < u_{n+1} < 1} dS_{[p]}(x)_{0,u_1} \otimes \dots \otimes dS_{[p]}(x)_{0,u_{n+1}} \right). \quad (5.9)$$

Hence, based on the definition of \mathcal{I}^n in (5.6), it can be proved inductively that (5.9) (so (5.8)) holds for all the elements in the step- $[p]$ nilpotent Lie group over any finite dimensional subspace of \mathcal{V} . Then by using continuity, we have (5.8) holds for any $a \in \mathcal{G}_{[p]}$.

Then we check that Y coincides with the rough integral in [25]. Indeed, based on (5.7), if we define $X : \{(s, t) | 0 \leq s \leq t \leq T\} \rightarrow 1 \oplus \mathcal{U} \oplus \dots \oplus \mathcal{U}^{\otimes [p]}$ by

$$X_{s,t} := 1 + \beta_s(g_s, g_{s,t}) + \sum_{n=2}^{[p]} \beta_s(g_s, \cdot)^{\otimes n} \mathcal{I}^{n-1}(g_{s,t}), \quad \forall 0 \leq s \leq t \leq T,$$

then X is a almost multiplicative functional (Def 3.1.1 [25]), and Y is a p -rough path associated with X (i.e. Y is multiplicative and there exist control ω and $\theta > 1$ such that $Y_{s,t}$ and $X_{s,t}$ are close up to an error bounded by $\omega(s, t)^\theta$ for any $s < t$). On the other hand, based on Def 3.2.2 and Theorem 3.2.1 [25], the rough integral is another p -rough path associated with X . Since the p -rough path associated with X is unique (Theorem 3.3.1 [25]), we have that Y coincides with the rough integral in [25]. ■

6 Iterated integration for weakly controlled paths

Recall in Section 3.1 that Condition 23 states that $T^{([p])}(\mathcal{V}) := \mathbb{R} \oplus \mathcal{V} \oplus \dots \oplus \mathcal{V}^{\otimes [p]}$ is the closure of the linear span of the topological group $\mathcal{G}_{[p]}$; Condition 25 requires the existence of a mapping \mathcal{I} which expresses the formal integral $\int \int_{0 < u_1 < u_2 < T} \delta g_{0,u_1} \otimes \delta g_{0,u_2}$ for $g \in C([0, T], \mathcal{G}_{[p]})$ as a universal continuous linear mapping of $g_{0,T}$. In particular, Conditions 23 and 25 are satisfied when $(T^{([p])}(\mathcal{V}), \mathcal{G}_{[p]}, \mathcal{P}_{[p]})$ is the triple for the step- $[p]$ free nilpotent Lie group $p \geq 1$ or the step-2 Butcher group $2 \leq p < 3$.

Corollary 48 (Iterated Integration for Weakly Controlled Paths) *Suppose $(T^{([p])}(\mathcal{V}), \mathcal{G}_{[p]}, \mathcal{P}_{[p]})$ satisfies Conditions 23 and 25, and $g \in C^{p-var}([0, T], \mathcal{G}_{[p]})$ for some $p \geq 2$. Suppose \mathcal{U}^i , $i = 1, 2$, are two Banach spaces, and there exist control ω and $\theta > 1$ such that $\gamma^i \in C([0, T], \mathcal{U}^i)$ and $\beta^i \in C([0, T], B(\mathcal{G}_{[p]-1}, \mathcal{U}^i))$, $i = 1, 2$, satisfy that*

$$\begin{aligned} M^i := & \sup_{0 \leq t \leq T} \|\beta_t^i(g_t, \cdot)\| + \sup_{0 \leq s < t \leq T} \max_{k=1, \dots, [p]-1} \frac{\|(\beta_t^i - \beta_s^i)(g_t, \cdot)\|_k}{\omega(s, t)^{\theta - \frac{k+1}{p}}} \\ & + \sup_{0 \leq s < t \leq T} \frac{\|\gamma_t^i - \gamma_s^i - \beta_s^i(g_s, g_{s,t})\|}{\omega(s, t)^{\theta - \frac{1}{p}}} < \infty, \quad i = 1, 2. \end{aligned} \quad (6.1)$$

Let us define $h \in C([0, T], \mathcal{U}^2 \oplus \mathcal{G}_{[p]})$ by $h := \gamma^2 \oplus g$ with $h_{s,t} = (\gamma_t^2 - \gamma_s^2) \oplus g_{s,t}$, $\forall 0 \leq s \leq t \leq T$. Then there exists $\beta \in C([0, T], B(\mathcal{U}^2 \oplus \mathcal{G}_{[p]}, \mathcal{U}^1 \otimes \mathcal{U}^2))$ such that (β, h) satisfies the integrable condition (Condition 14), and with control $\hat{\omega} := \omega + \|g\|_{p-var}^p$ and $\hat{\theta} := \min(\theta, \frac{[p]+1}{p}) > 1$, we have that $(\mathcal{I}$ in Condition 25)

$$\begin{aligned} & \left\| \int_s^t \beta_u(h_u) dh_u - (\gamma_s^1 - \gamma_0^1) \otimes (\gamma_t^2 - \gamma_s^2) - \beta_s^1(g_s, \cdot) \otimes \beta_s^2(g_s, \cdot) \mathcal{I}(g_{s,t}) \right\| \\ & \leq C_{p, \hat{\omega}(0, T)} M^1 M^2 \hat{\omega}(s, t)^{\hat{\theta}}, \quad \forall 0 \leq s < t \leq T. \end{aligned}$$

Remark 49 The integral path $\int_0^\cdot \beta(h) dh$ is continuous in p -variation w.r.t. M^i that is a norm involving both γ^i and β^i . It is hard to derive the continuity in operator norm of the one-form β in terms of γ^i and β^i (comparing with (4.2)), because the constraint $\|\gamma_t^2 - \gamma_s^2 - \beta_s^2(g_s, g_{s,t})\| \leq M^2 \omega(s, t)^{\theta - \frac{1}{p}}$ prescribes directions to evaluate β . The one-form β would not vary slowly as a linear operator (that induces strong continuity as demonstrated in Section 4); it only varies slowly in (a neighborhood of) the future direction of the path.

Proof. With the mapping $\mathcal{I} \in L(T^{([p])}(\mathcal{V}), T^{([p])}(\mathcal{V})^{\otimes 2})$ in Condition 25, for $s \in [0, T]$, we define $\beta_s^{1,2} \in C(\mathcal{G}_{[p]}, L(T^{([p])}(\mathcal{V}), \mathcal{U}^1 \otimes \mathcal{U}^2))$ by

$$\beta_s^{1,2}(a, v) := \beta_s^1(a, \cdot) \otimes \beta_s^2(a, \cdot) \mathcal{I}(v), \quad \forall a \in \mathcal{G}_{[p]}, \forall v \in T^{([p])}(\mathcal{V}),$$

and define $\beta \in C([0, T], B(\mathcal{U}^2 \oplus \mathcal{G}_{[p]}, \mathcal{U}^1 \otimes \mathcal{U}^2))$ by

$$\beta_s(u \oplus a, v \oplus b) := (\gamma_s^1 - \gamma_0^1) \otimes v + \beta_s^{1,2}(g_s, g_s^{-1}a(b-1)), \quad \forall (u \oplus a), (v \oplus b) \in \mathcal{U}^2 \oplus \mathcal{G}_{[p]}, \forall s \in [0, T].$$

Then we check that (β, h) satisfies the integrable condition. For $s < u < t$, similar to the argument used in the proof of the iterated integration for dominated paths (see (4.7) on page 25), we have

$$\begin{aligned} \|(\beta_u^{1,2} - \beta_s^{1,2})(g_u, \cdot)\|_k & \leq C_{p, \omega(0, T)} M^1 M^2 \omega(s, u)^{\theta - \frac{1}{p} - \frac{k}{p} + \frac{1}{p}} \\ & = C_{p, \omega(0, T)} M^1 M^2 \omega(s, u)^{\theta - \frac{k}{p}}, \quad \forall s < u, k = 1, \dots, [p]. \end{aligned}$$

Then,

$$\|(\beta_u^{1,2} - \beta_s^{1,2})(g_u, g_{u,t})\| \leq \sum_{\sigma \in \mathcal{P}_{[p]}} \|(\beta_u^{1,2} - \beta_s^{1,2})(g_u, \cdot)\|_{|\sigma|} \|\sigma(g_{u,t})\| \leq C_{p, \omega(0, T)} M^1 M^2 \hat{\omega}(s, t)^{\hat{\theta}}. \quad (6.2)$$

Based on (4.8), for $s < u < t$,

$$\begin{aligned} \beta_s^{1,2}(g_s, g_{s,t}) & = \beta_s^{1,2}(g_s, g_{s,u}) + \beta_s^{1,2}(g_u, g_{u,t}) + \beta_s^1(g_s, g_{s,u}) \otimes \beta_s^2(g_u, g_{u,t}) \\ & \quad - \sum_{\sigma_i \in \mathcal{P}_{[p]}, |\sigma_1| + |\sigma_2| \geq [p]+1} \beta_s^1(g_s, \sigma_1(g_{s,u} - 1)) \otimes \beta_s^2(g_s, \sigma_2(g_{s,u}(g_{u,t} - 1))) \\ & \quad - \sum_{k=2}^{[p]} \sum_{\sigma_i \in \mathcal{P}_{[p]}, |\sigma_1| + |\sigma_2| \geq [p]+1-k} \beta_s^1(g_s, \sigma_1(g_{s,u}) \cdot) \otimes \beta_s^2(g_s, \sigma_2(g_{s,u}) \cdot) \mathcal{I}(\pi_k(g_{u,t})). \end{aligned}$$

Then

$$\begin{aligned} & (\beta_u - \beta_s)(h_u, h_{u,t}) \\ & = (\gamma_u^1 - \gamma_s^1 - \beta_s^1(g_s, g_{s,u})) \otimes (\gamma_t^2 - \gamma_u^2) + \beta_s^1(g_s, g_{s,u}) \otimes (\gamma_t^2 - \gamma_u^2 - \beta_u^2(g_u, g_{u,t})) \\ & \quad + \beta_s^1(g_s, g_{s,u}) \otimes (\beta_u^2 - \beta_s^2)(g_u, g_{u,t}) + (\beta_u^{1,2} - \beta_s^{1,2})(g_u, g_{u,t}) \\ & \quad + \sum_{\sigma_i \in \mathcal{P}_{[p]}, |\sigma_1| + |\sigma_2| \geq [p]+1} \beta_s^1(g_s, \sigma_1(g_{s,u})) \otimes \beta_s^2(g_s, \sigma_2(g_{s,u}(g_{u,t} - 1))) \\ & \quad + \sum_{k=2}^{[p]} \sum_{\sigma_i \in \mathcal{P}_{[p]}, |\sigma_1| + |\sigma_2| \geq [p]+1-k} \beta_s^1(g_s, \sigma_1(g_{s,u}) \cdot) \otimes \beta_s^2(g_s, \sigma_2(g_{s,u}) \cdot) \mathcal{I}(\pi_k(g_{u,t})). \end{aligned}$$

Then combined with (6.1), (6.2) and that $\|\sigma(g_{s,t})\| \leq \|g\|_{p-var, [s, t]}^{|\sigma|}$, (β, h) satisfies the integrable condition (Condition 14), and the local estimate follows from Theorem 15. ■

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